C. P. Fuell
Room 4

ANALYTIC GEOMETRY

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO · DALLAS
ATLANTA · SAN FRANCISCO

MACMILLAN & CO., LIMITED
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN CO. OF CANADA, LTD.
TORONTO

ANALYTIC GEOMETRY

BY

CLYDE E. LOVE, PH.D.

ASSOCIATE PROFESSOR OF MATHEMATICS
IN THE UNIVERSITY OF MICHIGAN

New York
THE MACMILLAN COMPANY

1923

All rights reserved

PRINTED IN THE UNITED STATES OF AMERICA

Copyright, 1923,
By THE MACMILLAN COMPANY.

Set up and electrotyped. Published, January, 1923.

Press of
J. J. Little & Ives Co.
New York

PREFACE

Analytic geometry is rich in problems that require a certain amount of independent thinking, yet are not too difficult for the average student. In most of these problems, the result itself is of little intrinsic importance; the value of the work lies in the training afforded in reasoning and concentration. If the student be required to think for himself, first obtaining a knowledge of simple fundamental methods, and then extending, adapting, and generalizing these methods as need arises, the subject possesses the very highest disciplinary value.

Nevertheless, analytic geometry is sometimes presented as a series of formulas, with exercises requiring, in the main, a mere substitution of numbers for letters in the formulas. Such a course can be "studied" with a minimum of mental effort, and with correspondingly trifling benefit.

In the present book, the aim is to avoid unintelligent, mechanical processes wherever possible, thus insuring the student's real understanding of the work. In many of the simpler cases the derivation of results, or the discovery of alternative methods of derivation, is required of the reader. The problems of the text appear in the exercises in many forms and aspects, demanding a variety of methods of attack. Where the student is likely to need help in getting started on a new class of problems, suggestive examples are worked in the text.

To an extent somewhat greater than customary, the emphasis is placed on geometry rather than on analysis. The meaning of this statement is illustrated by the remarks of the next three paragraphs.

Every teacher will agree that the beginner thinks most easily and clearly when the geometric interpretation of his thought is most vivid. For this reason prominence is given to methods which, while analytic in their execution, are based on geometric rather than purely analytic reasoning. For instance, in finding the equation of a circle satisfying three conditions, the student is encouraged to carry out analytically the construction of elementary geometry for the center and radius.

In the study of the conics, while an ample amount of numerical drill is given, much attention is paid to the interesting and valuable geometric problems that exist in such large numbers.

In most texts, while plane loci are defined geometrically, those of space are defined by the equations representing them. It seems much more natural and logical to follow the former plan throughout. In this book the quadric surfaces are given a geometric definition, the equation being derived in each case from the definition.

It may be worth while to mention that wherever the book departs in any degree from custom, the practicability of the presentation here given has been thoroughly tested in the class-room.

The author wishes to acknowledge his deep obligation to Professor A. L. Nelson, who has read the entire manuscript with great care, and has also helped with the proofreading.

Clyde E. Love.

Ann Arbor, December, 1922.

CONTENTS

PLANE ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COÖRDINATES

ART. PAGE

1. Position of a point on a line 1
2. Directed line segments 1
3. Correspondence of real numbers to points on a straight line . 2
4. Position of a point on a surface 3
5. Cartesian coördinates 3
6. Units 4
7. Distance between two points 5
8. Midpoint of a line segment 8
9. Slope of a line 9
10. Parallel and perpendicular lines 10
11. Area of a triangle 12
12. Application of analysis to elementary geometry . . . 14

CHAPTER II

CURVES

13. Constants; variables 16
14. The locus of an equation 17
15. Scope of the term "curve" 20
16. Symmetry 21
17. Tests for symmetry 22
18. Intercepts on the axes 23
19. Curve tracing 24
20. Consequences of the definition of locus 26
21. Factorable equations 27
22. Algebraic transformation of equations 28
23. Points of intersection of two curves 29
24. Importance of checks 30
25. Non-intersecting curves 31
26. Number of points of intersection 31

CHAPTER III

LOCI

ART. PAGE
27. Path of a moving point 34
28. Loci defined geometrically 38

CHAPTER IV

THE STRAIGHT LINE

29. Introduction 42
30. Line parallel to a coördinate axis 42
31. Line through a given point in a given direction: point-slope form 43
32. Line through two points 46
33. Slope form 46
34. The general equation of the first degree 48
35. Parallel and perpendicular lines 49
36. Equations containing arbitrary constants; families of curves 51
37. Straight line determined by two conditions 53
38. The intercept form 54
39. The normal form 56
40. Distance between parallel lines; distance of a point from a line 58
41. A formula for the distance of a point from a line . . . 60
42. Angle bisectors 61
43. Angle between two lines 63
44. Infinite roots of an algebraic equation 65

CHAPTER V

THE CIRCLE

45. Definitions; standard forms 68
46. General equation 69
47. Point circles; imaginary circles 70
48. Circle satisfying three conditions 72
49. Quadratic equations 75
50. Intersection of a line and a circle; of two circles . . . 77
51. Tangents to plane curves 79
52. The condition for tangency 79
53. Tangents having a given slope 81
54. Tangent at a given point of contact 83
55. Tangents through an external point 84

ART. PAGE
56. Chord of contact 87
57. Curves through the intersections of two given curves . . 89
58. Circle through the intersections of two given circles . . 90
59. Common chord 93
60. Radical axis 93
61. Length of tangent 94

CHAPTER VI

POLAR COÖRDINATES

62. Polar coördinates 98
63. Curve tracing in polar coördinates 99
64. Locus problems in polar coördinates 101
65. Transformation from one system to the other . . . 103

CHAPTER VII

THE CONIC SECTIONS

I. General Introduction

66. Definitions 105
67. Species of conics 106
68. General equation 106
69. Polar equation of a conic 107

II. The Parabola

70. Definitions 109
71. Parabola with vertex as origin: first standard form . . 110
72. Other standard forms 111
73. General equation 112

III. The Ellipse

74. Ellipse referred to its axes: first standard form . . . 114
75. Other standard forms 116
76. General equation 117

IV. The Hyperbola

77. Hyperbola referred to its axes: first standard form 119
78. Asymptotes 120
79. Other standard forms 121
80. General equation 122
81. Equilateral, or rectangular, hyperbola 123

V. Coördinate Transformations

ART. PAGE
82. Introduction 125
83. Translation of axes 125
84. Rotation of axes 126

VI. The General Equation of the Second Degree

85. Introduction 129
86. Removal of the product term 130
87. Test for the species of a conic 132
88. The rectangular hyperbola with asymptotes parallel to the coördinate axes 133
89. Curve tracing by composition of ordinates 134
90. Limiting forms 138
91. The pair of straight lines 139
92. Conic through five points 142
93. Parabola through four points 144

CHAPTER VIII

THE PARABOLA

94. Introduction 146
95. Similar and equal geometric figures 148
96. Similarity and equality of parabolas 148
97. Geometric problems involving the parabola 149
98. Tangent at a given point 154
99. Normal; subtangent; subnormal 156
100. Subtangent and subnormal for the parabola 156
101. Tangents having a given slope; tangents drawn from an external point 159
102. Number of tangents through a given point 160
103. Two geometric properties 161
104. Chord of contact 163
105. Loci by elimination of parameters 164
106. Diameter of a conic 167
107. Diameter of a parabola 167
108. Geometric properties involving diameters 169

CHAPTER IX

THE CENTRAL CONICS

109. Introduction 173
110. A new definition of the ellipse 175
111. A new definition of the hyperbola 177

ART. PAGE
112. Geometrical construction of the central conics 177
113. Tangent at a given point of contact 179
114. A property of the tangent to a central conic 180
115. Tangents having a given slope 181
116. Another property involving tangents 184
117. Auxiliary circles; eccentric angle 186
118. Parametric equations 187
119. Parametric equations of the ellipse 188
120. Parametric equations of the hyperbola 188
121. Chord of contact 191
122. Diameters 191
123. Geometric properties involving diameters 193
124. Points at infinity; curves intersecting at infinity . . . 195
125. Number of intersections at infinity 196
126. Test for the species of a conic 197
127. Asymptotes 200
128. Asymptotes of the hyperbola 200
129. Single equation representing the pair of asymptotes . . 201
130. A general method for determination of asymptotes . . 204
131. Equilateral hyperbola referred to its asymptotes . . . 206

SOLID ANALYTIC GEOMETRY

CHAPTER X

COÖRDINATES IN SPACE

132. Rectangular cartesian coördinates 208
133. Figures 209
134. Distance between two points 210
135. Midpoint of a line segment 210
136. Direction angles; direction cosines 213
137. Direction cosines proportional to three numbers . . . 214
138. Radius vector of a point 215
139. Direction cosines of the line through two points . . . 215
140. Projections 218
141. Angle between two lines 218
142. Perpendicular lines 219
143. Cylindrical coördinates 221

CHAPTER XI

LOCI

144. The locus of an equation 222
145. Determination of points on a surface; intercepts on the axes 223

ART. PAGE
146. Planes parallel to a coördinate plane 223
147. Plane sections of a surface; traces 223
148. The locus of two simultaneous equations 225
149. Determination of points on a curve; piercing points . . 226
150. Intersection of three surfaces; of a curve and a surface . 227
151. The surface as locus of a moving point 228
152. The curve as locus of a moving point 228

CHAPTER XII

THE PLANE

153. Normal form 230
154. General form; reduction to normal form 231
155. Perpendicular line and plane 232
156. Parallel planes 234
157. Distance between parallel planes; between a point and a plane 235
158. Plane through a given point 236
159. Plane determined by three points 236
160. Angle between two planes 238
161. Perpendicular planes 238
162. Planes perpendicular to a coördinate plane 239
163. Intercept form 240
164. Intersection of three planes 241

CHAPTER XIII

THE STRAIGHT LINE

165. General form 242
166. Planes through a given line 242
167. Projecting planes 243
168. Symmetric form 245
169. Determination of direction cosines; reduction to the symmetric form 246
170. Reduction to the symmetric form: second method . . 247
171. Line parallel to a coördinate plane 247
172. Parallel, intersecting, and skew lines 248
173. Perpendicular line and plane 250
174. Angle between a line and a plane 251
175. Parallel line and plane 251

CHAPTER XIV

SURFACES

ART. PAGE
176. Introduction 255
177. Symmetry 255
178. Sketching by parallel plane sections 256
179. Generation of a surface by a moving curve 258
180. Surfaces of revolution 258
181. The sphere 263
182. Tangent plane; tangent lines 263
183. Quadric surfaces 265
184. The ellipsoid 266
185. The hyperboloid of one sheet 268
186. The hyperboloid of two sheets 270
187. The elliptic paraboloid 273
188. The hyperbolic paraboloid 274
189. Ruled surfaces 277
190. The hyperboloid of one sheet as a ruled surface . . . 278
191. The hyperbolic paraboloid as a ruled surface . . . 279
192. Cylinders 282
193. Equations in two variables: cylinders perpendicular to a coördinate plane 283
194. Quadric cylinders 284
195. The equation of a cylinder: general case 284
196. Cones 287
197. The elliptic cone 287
198. The general equation of the second degree 291

CHAPTER XV

CURVES

199. Surfaces through a given curve 294
200. Projecting cylinders 294
201. Plane sections of a quadric surface 298
202. Plane sections of a quadric cylinder 299
203. The conic sections 300

INDEX 303

ANALYTIC GEOMETRY

PLANE ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COÖRDINATES

1. Position of a point on a line. If a point is known to lie on a given line, its position is usually determined by means of its distance from some fixed point of reference on the line. For instance, Ann Arbor is on the Michigan Central Railroad 37 miles west of Detroit; in a Fahrenheit thermometer the freezing point of water is 32° above zero. It is clear that in order to fix the position of the point completely, we must know not merely its distance from the point of reference, but also the direction, or *sense*, in which that distance is measured. In the examples just cited, the sense of measurement is expressed by the words "west of" and "above," respectively.

2. Directed line segments. As suggested by the examples of § 1, it is frequently necessary to consider a line segment as measured in a definite sense *from* one endpoint *to* the other. When this is done, the segment is said to be *directed*. If the terminal points are A, B, we speak of the segment AB or the segment BA according as the sense is from A to B or from B to A.

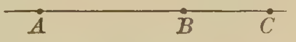


FIG. 1

If one sense is chosen as positive, then the opposite sense is negative: thus

$$AB = -BA,$$

or

$$AB + BA = 0.$$

If C is any third point of the straight line determined by A and B, then for all possible positions of A, B, and C we have

$$AB + BC = AC, \tag{1}$$

or

$$AB + BC + CA = 0. \tag{2}$$

The student should test these formulas thoroughly, taking the points A, B, C in various relative positions on the line.

Two directed line segments lying in the same line or in parallel lines are said to be *equal* if they have the same length and are measured in the same sense.

The ordinary undirected segment joining two points A, B will be denoted by either of the symbols AB, BA. In cases where doubt may arise as to whether the segment is directed or undirected, an explicit statement will be made.

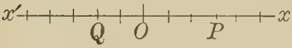


FIG. 2

3. Correspondence of real numbers to points on a straight line. Given a point P lying on the straight line $x'x$ (Fig. 2), let us choose an arbitrary fixed point O on the line, and determine the position of P by its directed distance from O, measured in some suitable unit.

If now we agree that distances measured to the right shall be called positive, those to the left negative, then the position of P may be indicated completely by giving the magnitude of OP with the proper sign prefixed. (When the distance is positive, the sign is usually omitted.) Thus in the figure P is at the distance 3, Q at the distance -2, from O. In this way there is built up a *correspondence* between the real numbers of algebra and the points of the straight line: *to every real number there corresponds one and only one point of the line, and conversely.*

The directed segment OP, represented algebraically by a number, is called the *coördinate*, or the *abscissa*, of P, and the fixed point O is called the *origin*.

4. Position of a point on a surface. If a point lies on a given surface, two magnitudes, or coördinates, are necessary to determine its position. Thus we say that one town is ten miles east and eight miles north of another; the position of a picture on a wall may be given by its height above the floor and its distance from one corner of the room; in geography, the location of a point is indicated by its latitude and longitude. Here again it is clear that each coördinate must be measured in a definite sense.

5. Cartesian coördinates. Given a point P lying in a certain plane, let us assume two perpendicular lines $x'x$, $y'y$ lying in the plane. The line $x'x$ is called the x-axis, $y'y$ the y-axis, and their point of intersection O is the *origin.* The position of P is evidently known if its distances from the axes are given, each being measured in a definite sense. These directed segments are called the *cartesian coördinates* * of P: the distance from the y-axis — NP or its equal OM — is the *abscissa*, the distance MP from the x-axis is the *ordinate.* As in § 3, each coördinate is represented algebraically by a number.

We shall ordinarily assume the axes as in Fig. 3, and shall consider abscissas *positive* if measured *to the right, negative to the left;* ordinates *positive upward, negative downward.*

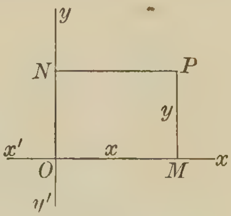


Fig. 3

It is customary to write the coördinates of a point in parenthesis, with the abscissa first: thus the point P: (3, 5), also written simply (3, 5), has the abscissa 3 and the ordinate 5 (the point P in

---

* More precisely, *rectangular* cartesian coördinates. The axes are sometimes taken oblique to each other, but in this book only the rectangular system will be used.

The word *cartesian* is derived from the name of René Descartes (1596–1650), who was the founder of analytic geometry.

Fig. 4). The figure also shows the points Q: $(2, -4)$, R: $(-4, -3)$, and S: $(-3, 0)$.

The axes divide the plane into four compartments, called *quadrants*, and numbered as in Fig. 4. Thus the abscissa is positive in the first and fourth quadrants, the ordinate positive in the first and second.

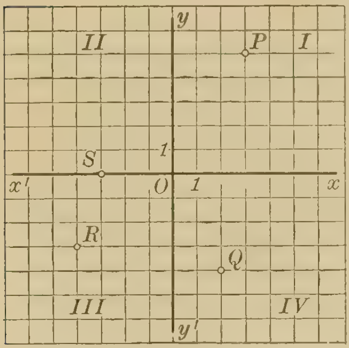


FIG. 4

By the introduction of a cartesian coördinate system there is set up a unique correspondence between *points*, on the one hand, and *pairs of real numbers*, on the other: to every pair of real numbers there corresponds one and only one point in the plane, and conversely.

In all work with cartesian coördinates, unless the contrary is explicitly stated it will be assumed that segments oblique to the axes are undirected, segments parallel to an axis are directed. Segments parallel to Ox will be considered positive if measured to the right, negative to the left; segments parallel to Oy, positive upward, negative downward.

6. Units. In analytic geometry, drawings are usually made on square-ruled paper, called *coördinate paper* (see Fig. 4). The unit of measurement chosen need not, and usually should not, be the width of one space on the coördinate paper; the scale should be selected with regard to the nature of the drawing to be made — neither so large that some of the points fall beyond the limits of the paper, nor so small that the properties of the figure become obscured. Thus if the points to be plotted are $(20, 45)$, $(36, -13)$, etc., one space on the paper might represent 4 units, while if the points are $(0.1, 0.3)$, $(-0.25, 0)$, etc., we might conveniently let 20 spaces represent the unit. The scale adopted should always be clearly indicated.

Cases sometimes arise in which it is convenient to adopt different scales on the two axes; but unless the contrary is indicated we shall assume throughout that the unit for ordinates is the same as that for abscissas.

To plot a point whose coördinates are irrational, we approximate as closely to the irrational numbers involved as our scale will allow. For instance, to plot the point $(\sqrt{2}, \sqrt{3})$, we would in most cases take

$$\sqrt{2} = 1.4, \ \sqrt{3} = 1.7,$$

but on a very large scale it might be possible to put

$$\sqrt{2} = 1.41, \ \sqrt{3} = 1.73.$$

7. Distance between two points. In the remainder of this chapter we develop a few elementary formulas which arise directly from the definition of cartesian coördinates, and which will be of frequent service in our later work.

The distance between two points P_1, P_2 can be expressed in terms of their coördinates by the theorem of Pythagoras. Let the coördinates of the two points be denoted by the letters x, y with subscripts: P_1: (x_1, y_1), P_2: (x_2, y_2). Now, in Fig. 5,

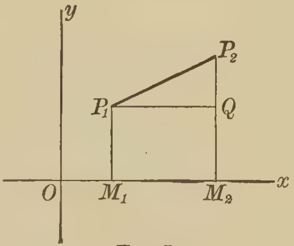


FIG. 5

$$d = \sqrt{\overline{P_1Q}^2 + \overline{QP_2}^2};$$

but

$$P_1Q = M_1M_2 = OM_2 - OM_1 = x_2 - x_1,$$
$$QP_2 = M_2P_2 - M_2Q = y_2 - y_1,$$

whence

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1}$$

By drawing the figure in various positions, the student may convince himself that the formula holds no matter where the points P_1, P_2 may be situated — not merely when they happen to lie in the first quadrant.

Examples: (*a*) Find the distance between the points (3, 2) and (– 5, 4).

By formula (**1**), or directly from a figure, we find

$$d = \sqrt{(-5-3)^2+(4-2)^2} = \sqrt{68} = 2\sqrt{17}\,.$$

(*b*) A point moves so that its distance from the origin is always 2. Express this fact by means of an algebraic equation.

Let the coördinates of the moving point be (x, y). Then, by (**1**), the distance of this point from the origin is $\sqrt{x^2+y^2}$. Hence the required equation is

$$\sqrt{x^2+y^2} = 2,$$

or

$$x^2+y^2 = 4.$$

The locus of the point (x, y) is evidently a circle of radius 2 with center at O.

EXERCISES

1. Verify formula (1), § 2, (*a*) with C lying between A and B; (*b*) with C lying in BA produced.

2. Show that formulas (1) and (2) of § 2 can be extended to apply to any number of points.

Plot the following points, choosing a suitable scale in each exercise.

3. $(2, 4)$, $(-3, 6)$, $(0, 5)$, $(-8, -1)$, $(\frac{7}{3}, -\frac{17}{6})$, $(-2\sqrt{2}, 2\sqrt{5})$.

4. $(\frac{3}{2}, \frac{2}{3})$, $(-\frac{1}{10}, \frac{8}{5})$, $(-\frac{7}{5}, \frac{7}{8})$, $(\sqrt{\frac{1}{3}}, \frac{1}{5}\sqrt{10})$.

5. $(-10, -8)$, $(35, 23)$, $(30, -3)$, $(-15, -25)$.

6. $(\frac{1}{2}\sqrt{2}, 0)$, $(-2\sqrt{3}, \sqrt{3})$, $(-\frac{1}{2}, \frac{1}{2}\sqrt{6})$, $(\pi, \frac{1}{2}\pi^2)$.

7. Draw the quadrilateral whose vertices are $(0.07, 0.11)$, $(-0.03, 0.06)$, $(0, 0.13)$, $(0.20, -0.08)$.

8. What can be said of the coördinates of all points on the x-axis? On the y-axis? On the line through O bisecting the first and third quadrants? The second and fourth quadrants? On the line parallel to the x-axis two units to the right of it? On the line parallel to the y-axis three units below?

Find the distances between the following pairs of points.

9. $(2, 4)$, $(3, 7)$. *Ans.* $\sqrt{10}$.

10. $(-2, -5)$, $(4, -6)$. *Ans.* $\sqrt{37}$.

11. $(3, -2)$, $(-5, -8)$.

12. $(\frac{3}{2}, -\frac{5}{4})$, $(-2, \frac{7}{6})$. *Ans.* $\frac{1}{12}\sqrt{2605}$.

13. $(-\frac{3}{5}, -\frac{9}{5})$, $(\frac{1}{10}, -\frac{3}{20})$.

14. (x, y), (h, k).

15. Given that the point (x, y) is at the distance 5 from the point $(-5, 3)$, express this fact by means of an equation. What is the locus of such points? Draw the figure.

16. Express by an equation the statement that the point (x, y) is equidistant from the points $(3, 4)$ and $(-1, -2)$. What is the locus of such points? Draw the figure. *Ans.* $2x + 3y = 5$.

17. Express by an equation the fact that the point (x, y) is equidistant from the points $(3, 2)$ and $(-5, 2)$. *Ans.* $x + 1 = 0$.

18. Derive the distance formula (**1**) from a figure in which P_2 is below and to the right of P_1.

19. Derive the distance formula, taking P_1 in the third and P_2 in the first quadrant.

20. Derive the distance formula, taking P_1 in the fourth and P_2 in the second quadrant.

21. Prove that the triangle whose vertices are $(2, -2)$, $(-1, -1)$, $(1, 5)$ is a right triangle, and find its area.

22. Prove that the triangle whose vertices are $(2, 3)$, $(-3, 0)$, and $(5, 8)$ is isosceles, and find its area. *Ans.* $A = 8$.

23. Prove that the points $(-3, 8)$, $(-7, 6)$, $(-3, -2)$, $(1, 0)$ are the vertices of a parallelogram. Is the parallelogram a rectangle?

24. Prove that the points $(0, -1)$, $(2, 1)$, $(0, 3)$, $(-2, 1)$ are the vertices of a square.

25. Find the radius of the circle whose center is $(2, 5)$ and which passes through $(14, 10)$. Does this circle also pass through $(13, 12)$?

26. A circle touches the y-axis, and its center is at $(5, 6)$. Does the circle pass through the point $(4, 1)$? Through $(1, 3)$?

27. Find the radius of a circle with center at $(3, 0)$, if a chord of length 4 is bisected at $(5, 4)$. *Ans.* $2\sqrt{6}$.

28. The points $(10, -2)$ and $(-4, -4)$ are the ends of a diameter of a circle. Does this circle pass through $(-2, 2)$?

29. The center of a circle is the point $(-4, 2)$ and its radius is 5. Find the length of that chord which is bisected at $(-2, 1)$.

30. Prove that the points $(10, 2)$, $(7, 1)$, $(-2, -2)$ lie in a straight line.

31. Determine whether the points $(3, 0)$, $(-1, 8)$, $(48, -90)$ lie in a straight line.

32. Do the points $(1, 8)$, $(-3, -4)$, $(6, 24)$ lie in a straight line?

33. Do the points $(\frac{1}{3}, \frac{1}{3})$, $(\frac{3}{4}, \frac{1}{8})$, $(-\frac{1}{2}, \frac{3}{4})$ lie in a straight line?

34. A point is at the distance 4 from the origin and 2 from the x-axis. Find its coördinates.

35. A point is at the distance 10 from the origin and -6 from the y-axis. What are its coördinates?

36. Prove that if the coördinates of the endpoints of a line segment are rational numbers, the square on that segment contains a rational number of square units. Is the converse true?

8. Midpoint of a line segment. Let $P: (x, y)$ be the point midway between the points $P_1: (x_1, y_1)$ and $P_2: (x_2, y_2)$. In Fig. 6,

$$OM = OM_1 + M_1M.$$

But

$$OM = x, OM_1 = x_1,$$

while

$$M_1M = \tfrac{1}{2} M_1M_2 = \tfrac{1}{2}(x_2 - x_1).$$

Hence

$$x = x_1 + \tfrac{1}{2}(x_2 - x_1).$$

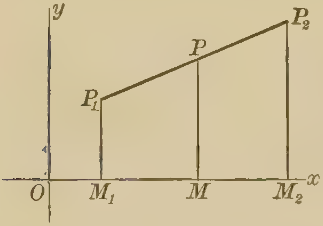


FIG. 6

Similarly,

$$y = y_1 + \tfrac{1}{2}(y_2 - y_1).$$

Simplifying, we find

$$x = \tfrac{1}{2}(x_1 + x_2), \quad y = \tfrac{1}{2}(y_1 + y_2), \tag{2}$$

or in words, *the coördinates of the midpoint of a line segment are the arithmetic means of the coördinates of the endpoints.*

It is obvious that these formulas can be used not merely to find the midpoint when the endpoints are given, but also to find one endpoint when the other endpoint and the midpoint are given, as in the following

Example: The line from (3, 2) to (5, – 6) is produced to a point P so that its length is doubled. Find the coördinates of P.

Here one endpoint is (3, 2), the midpoint is (5, – 6), while the other endpoint (x_2, y_2) is to be found.

Substituting in (**2**), we have

$$5 = \tfrac{1}{2}(3 + x_2), \quad -6 = \tfrac{1}{2}(2 + y_2),$$

whence

$$x_2 = 7, \; y_2 = -14.$$

Thus P is the point (7, – 14).

9. Slope of a line. The *slope* of a straight line is defined as the *tangent of the angle made by the line with the positive x-axis*, this angle being measured *from Ox to* the line. Slope is usually denoted by the letter m.

Slope-angles are considered positive if measured from the positive x-axis toward the positive y-axis: i.e. if the axes are in the conventional position, a slope-angle is *positive* if measured *counterclockwise*, *negative* if measured *clockwise*. Thus lines sloping upward to the right have positive slope, those sloping downward to the right have negative slope.

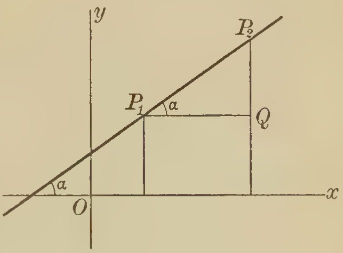


FIG. 7

The slope of the line joining the points P_1: (x_1, y_1) and P_2: $(x_2\, y_2)$ is evidently

$$m = \tan\alpha = \frac{QP_2}{P_1Q},$$

whence

(**3**)
$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

It should be noted that the idea of slope becomes meaningless in the case of a line parallel to the y-axis, since $\tan 90° = \infty$.

10. Parallel and perpendicular lines. If two lines are parallel, they evidently have the same slope; and conversely.

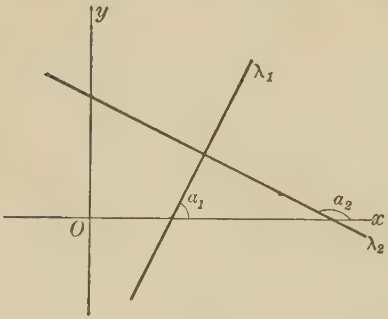


FIG. 8

Given two perpendicular lines λ_1, λ_2, the slope of λ_1 is

$$m_1 = \tan \alpha_1,$$

that of λ_2 is

$$m_2 = \tan \alpha_2.$$

But $\alpha_2 = \alpha_1 + 90°$, so that, by trigonometry,

$$\tan \alpha_2 = -\cot \alpha_1 = -\frac{1}{\tan \alpha_1},$$

or

$$m_2 = -\frac{1}{m_1}. \tag{4}$$

Thus *if two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other.* The same theorem is sometimes expressed in the form: *If two lines are perpendicular, the product of their slopes is* -1.

Here also the converse is true, as may easily be shown.

EXERCISES

Find the point midway between each of the following pairs of points. Check by plotting.

1. $(1, 2)$, $(3, 5)$.

2. $(-1, -3)$, $(-4, 2)$.

3. $(\frac{3}{2}, -\frac{5}{3})$, $(-\frac{4}{3}, -\frac{5}{2})$.

4. $(-\frac{1}{8}, -\frac{3}{8})$, $(\frac{4}{5}, -\frac{2}{5})$.

5. To what point must the segment from $(-1, 4)$ to $(-5, -9)$ be extended, if its length is to be doubled? Trebled? Quadrupled?

6. The center of a circle is at $(5, 5)$; one point of the circle is $(6, -2)$. Find the other end of the diameter through this point.

7. Three vertices of a rectangle are (3, − 2), (7, 6), (3, 8). Find the fourth vertex.

8. Three vertices of a parallelogram are (5, 2), (− 2, − 3), (3, − 4). Find the fourth vertex. Check by plotting. How many answers are there?

Find the slopes of the lines joining the following pairs of points.

9. (2, − 2), (− 5, 1). **10.** (3, 3), (− 6, − 2).

11. $(-\frac{4}{5}, \frac{3}{10})$, $(-\frac{1}{10}, \frac{1}{5})$. **12.** $(\frac{2}{7}, \frac{12}{7})$, $(-\frac{3}{4}, \frac{3}{8})$.

13. Find the slope of the line from (− 3, 5) to (− 3, − 2). What does the result mean?

14. Show by two new methods that the triangle whose vertices are (2, − 2), (− 1, − 1), (1, 5) is a right triangle. (Ex. 21, p. 7.)

15. Prove in a new way that the points (2, 3), (− 3, 0), (5, 8) form an isosceles triangle. (Ex. 22, p. 7.)

16. Prove by several methods that the points (2, 2), (3, 6), (5, − 1), (4, − 5) form a parallelogram.

17. Prove in a new way that the points (0, − 1), (2, 1), (0, 3), (− 2, 1) form a square. (Ex. 24, p. 7.)

18. Prove in three ways that the point (14, 6) lies on the circle having the points (6, − 2), (10, 10) as ends of a diameter.

19. Prove by a new method that the points (10, 2), (7, 1), (− 2, − 2) lie in a straight line. (Ex. 30, p. 8.)

20. Prove in two ways that the points (6, − 2), (10, 10), (11, 13) lie in a straight line.

21. Show that the points (− 5, − 3), $(2, \frac{2}{3})$, (7, − 6), (1, − 11) are the vertices of a trapezoid.

22. Show in two ways that the perpendicular bisector of the line from $(\frac{1}{2}, -\frac{3}{2})$ to $(\frac{3}{4}, -\frac{1}{4})$ passes through $(\frac{1}{4}, -\frac{4}{5})$.

23. Prove that the points $(\frac{17}{5}, \frac{6}{5})$, $(\frac{33}{5}, -\frac{6}{5})$, $(\frac{32}{5}, -\frac{24}{5})$, (0, 0) are the vertices of an isosceles trapezoid.

24. Find the area of the trapezoid of Ex. 23.

25. Derive the midpoint formulas (**2**), using a figure in which P_1 lies in the second and P_2 in the fourth quadrant.

26. Derive the midpoint formulas with P_1 and P_2 both lying in the third quadrant.

27. Derive formula (**3**), § 9, with P_1 in the first and P_2 in the third quadrant.

28. Derive formula (**3**) with both points in the second quadrant.

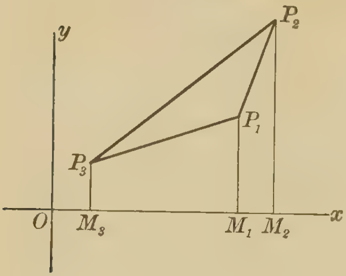


FIG. 9

11. Area of a triangle. In Fig. 9, if from the area of the trapezoid $M_3M_2P_2P_3$ we subtract the areas of the trapezoids $M_3M_1P_1P_3$ and $M_1M_2P_2P_1$, there remains the area of the triangle $P_1P_2P_3$. Let the points P_1, P_2, P_3 be (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively. Then

$$M_3M_2P_2P_3 = \tfrac{1}{2}(x_2 - x_3)(y_2 + y_3),$$

$$M_3M_1P_1P_3 = \tfrac{1}{2}(x_1 - x_3)(y_1 + y_3),$$

$$M_1M_2P_2P_1 = \tfrac{1}{2}(x_2 - x_1)(y_2 + y_1).$$

Thus

$$A = \tfrac{1}{2}[(x_2 - x_3)(y_2 + y_3) - (x_1 - x_3)(y_1 + y_3) - (x_2 - x_1)(y_2 + y_1)],$$

or after rearranging,*

$$A = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \tag{5}$$

The area of a given triangle as found by this formula may be either positive or negative, according to the order in which the points are taken. We shall in general consider area as essentially positive, using the formula to find the number of square units regardless of sign.

EXERCISES

Find the areas of the triangles having the following vertices.

1. $(0, 0)$, $(6, 1)$, $(3, -2)$.

2. $(4, -1)$, $(-2, -3)$, $(5, 5)$.

3. $(2, 6)$, $(5, 6)$, $(-1, -4)$.

4. $(-7, 5)$, $(3, -8)$, $(8, 4)$.

5. $(\frac{1}{3}, \frac{2}{3})$, $(-\frac{1}{6}, \frac{1}{2})$, $(-\frac{4}{3}, -\frac{7}{6})$.

6. $(38, 16)$, $(41, 17)$, $(-36, -14)$.

---

* The student who is familiar with the use of third-order determinants will note that (5) may be written in the more easily remembered form

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \tag{5'}$$

7. Find the area of the quadrilateral whose vertices are (1, 1), (2, 5), (4, 3), (4, − 6). *Ans.* $\frac{37}{2}$.

8. Find the area of the quadrilateral whose vertices are (2, 1), (5, 3), (7, 9), (6, − 5). Check by solving in two ways.

9. Prove in another way that the points (10, 2), (7, 1), (− 2, − 2) lie in a straight line. (Ex. 30, p. 8.)

10. Prove in three ways that the points (4, 2), (6, − 3), (10, − 13) lie in a straight line.

11. Find the distance of the point (3, 2) from the line joining the points (− 1, 2), (1, − 6). *Ans.* $\frac{16}{17}\sqrt{17}$.

12. Find the distance of the point (− 3, 3) from the line joining the points (2, 7), (1, − 2).

13. Prove in a new way that the points (2, − 2), (− 1, − 1), (1, 5) form a right triangle. (Ex. 21, p. 7.)

14. Prove in a new way that the points (0, − 1), (2, 1), (0, 3), (− 2, 1) are the vertices of a square. (Ex. 24, p. 7.)

15. Derive formula (5) by dropping perpendiculars from P_1, P_2, P_3 to the y-axis.

16. Three of the vertices of a rectangle are (2, 2), (6, 5), (4, 6). Find the area by two methods.

17. Find the fourth vertex of the rectangle of Ex. 16.

18. Three vertices of a parallelogram are (− 2, 0), (− 7, 4), (3, 3). Find the area.

19. Find the fourth vertex of the parallelogram of Ex. 18.

20. Prove that the area of a parallelogram three of whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by the formula

$$A = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2).$$

21. Prove that if the coördinates of the vertices of a parallelogram are whole numbers, the parallelogram contains a whole number of square units. Is this theorem true for any polygon?

22. Prove that if the coördinates of the vertices of a polygon are rational numbers, the area of the polygon is also a rational number.

23. Show that it is impossible to construct an equilateral triangle the coördinates of whose vertices are all rational numbers. (Cf. Ex. 22 above and Ex. 36, p. 8.)

12. Application of analysis to elementary geometry. The representation of points by pairs of numbers, which is effected by means of a cartesian coördinate system, establishes a connection between algebra and geometry which enables us to give geometric problems an algebraic setting, and to solve them by purely algebraic means. As we proceed, this connection between the two sciences will be developed and elaborated in various directions.

Many important theorems of elementary geometry, particularly those involving polygons, can be proved analytically by the theory of the present chapter; the process may be indicated by an

Example: Prove that the diagonals of a parallelogram bisect each other.

Let us place the parallelogram in the position shown. The coördinates of O are $(0, 0)$, those of P_1 may be taken as $(x_1, 0)$, those of P_2 as (x_2, y_2), whence those of P_3 *must be* $(x_1 + x_2, y_2)$. Thus the midpoint of P_1P_2 is $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$; likewise the midpoint of OP_3 is $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$, which proves the theorem.

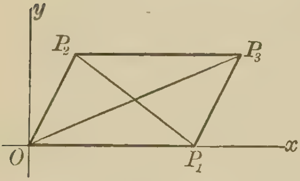


FIG. 10

It is obvious that *if the property to be proved is independent of the position of the figure, there is no loss of generality in assuming the figure in any convenient position* with reference to the coördinate axes. Thus the proof just given is entirely general. On the other hand, the figure itself must not be made special in any way: for instance, in the above example, by assigning numerical coördinates to the vertices, or by choosing as the given parallelogram a rhombus or a rectangle.

Further, this point must be carefully noted. While in elementary geometry the proof is obtained by studying the properties of the figure, in analysis the proof arises from algebraic work involving the coördinates: the figure is of no use at all beyond helping us to bear in mind the nature of the problem. Our notation must therefore be such that *the coördinates themselves express the data* of the problem. In the example, when coördinates have been assigned to three of the vertices, those of the fourth are determined, and must be correctly expressed in terms of those already assigned. If this is not done, the proof is impossible.

EXERCISES

Prove the following theorems.

1. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

2. The diagonals of a rectangle are equal.

3. The distance between the midpoints of the non-parallel sides of a trapezoid is half the sum of the parallel sides.

4. An isosceles triangle has two equal medians.

5. A triangle having two equal medians is isosceles.

6. The diagonals of a rhombus are perpendicular.

7. If the diagonals of a rectangle are perpendicular, the rectangle is a square.

8. If a convex quadrilateral has two opposite sides equal and parallel, it is a parallelogram.

9. The lines joining the midpoints of the sides of a triangle divide it into four equal triangles.

10. A quadrilateral whose diagonals bisect each other is a parallelogram.

11. A quadrilateral whose diagonals bisect each other at right angles is a rhombus.

12. The diagonals of an isosceles trapezoid are equal.

13. A trapezoid whose diagonals are equal is isosceles.

14. The line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

15. The line segments joining the midpoints of adjacent sides of a quadrilateral form a parallelogram.

CHAPTER II

CURVES

13. Constants; variables. In algebra (and to some extent even in arithmetic) we become familiar with the use of letters to represent numerical quantities. Two kinds of quantities occur in algebra: the knowns, usually represented by a, b, $\cdots$, and the unknowns, represented by x, y, $\cdots$. Thus the general quadratic equation is written in the form

$$ax^2 + bx + c = 0;$$

the equation of the first degree in three unknowns is written

$$ax + by + cz + d = 0.$$

In analytic geometry the quantities with which we have to deal are "constants" and "variables."

A *constant* is a quantity whose value remains unchanged throughout any given problem. Examples are the coördinates of a fixed point, the radius of a circle, the slope of a line, etc. Coördinates of fixed points are denoted by the letters x, y with subscripts, as (x_1, y_1), (x_2, y_2), etc.; other letters, particularly those near the beginning of the alphabet, are also used to denote constants.

A *variable* is a quantity that may take different values (usually an infinite number of them) in the same problem. The variables most frequently occurring are the coördinates x, y of a point that is free to assume various positions — for instance by moving along a definite path.

For example, the fact that the point (x, y) is at the constant distance a from the origin is expressed by the equation

$$\sqrt{x^2 + y^2} = a,$$

or

$$x^2 + y^2 = a^2.$$

This equation evidently remains true if the point (x, y) moves along the circle of radius a with center at the origin. As the point moves, both coördinates vary, but the radius remains the same.

Again, if the point (x, y) is equidistant from the points $(0, 0)$ and (x_1, y_1), we have

$$\sqrt{x^2 + y^2} = \sqrt{(x - x_1)^2 + (y - y_1)^2},$$

which upon squaring and simplifying becomes

$$2\,x_1x + 2\,y_1y = x_1^2 + y_1^2.$$

Here the point (x, y) may take any position on the perpendicular bisector of the line joining the fixed points $(0, 0)$ and (x_1, y_1).

14. The locus of an equation. Let two variables x and y be connected by an equation — for instance

$$y = 2\,x + 3,$$

$$y^3 = 2\,x,$$

or

$$x^2 + y^2 = 1.$$

If any value be assigned to either variable, the corresponding value or values of the other may be found at once from the equation; thus there exist in general infinitely many pairs of values of x and y that satisfy the equation. Each pair of numbers may be represented geometrically by a point. The points so determined are not scattered at random throughout the plane, but form in the aggregate a definite *curve*, as may be verified for any given equation by plotting any number of the points. This curve is called the *locus* of the equation. The definition, which is fundamental for our future work, may be stated more concisely as follows:

The locus of an equation is a curve containing those points, and only those points, whose coördinates satisfy the equation.

The curve corresponding to a given equation is said to *represent the equation geometrically,* while the equation *represents the curve analytically.* The study of plane curves (i.e. curves all of whose points lie in a plane) by means of the equations representing them forms the subject-matter of plane analytic geometry.

The elementary method of tracing the curve representing a given equation is to find a number of pairs of values of x and y satisfying the equation, plot the corresponding points, and draw a smooth curve through them: this curve is approximately the desired locus. The method of point-plotting frequently involves a good deal of tedious numerical work, but the great objection is that it sheds but little light on the general properties of the curve; it is, however, the only method available at the present stage of our work. The additional theory to be developed in this and later chapters will enable us to trace a variety of important curves with a comparatively small amount of point-plotting.

If coördinate paper be used, only two instruments are required to make a neat and reasonably accurate sketch: a straight edge, to put in the axes and any other lines that may occur, and a small "irregular curve" such as draftsmen use, to draw a smooth curve through the plotted points. By the use of these instruments much better results are obtained than by free-hand drawing.

Examples: (*a*) Trace the locus of the equation

$$y = 2x + 3.$$

It will be most convenient to assign values to x and compute the values of y:

x	0	1	2	3	-1	-2	-3	-4
y	3	5	7	9	1	-1	-3	-5

We now plot these points, choosing in this case one space on the coördinate paper as the unit, and draw a smooth curve through them. (It should be noted that the curve picks up the points *according to the algebraic order of the values of* x — not as they are arranged in the above table.) The "curve" in this case appears to be a straight line: it will be proved in § 34, and assumed in the meanwhile, that *the locus of every equation of the first degree is a straight line.*

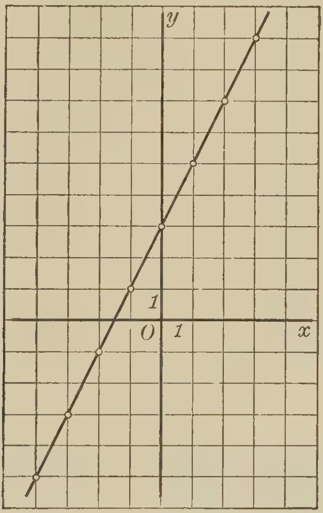


FIG. 11

(*b*) Plot the curve

$$y^3 = 2\,x.$$

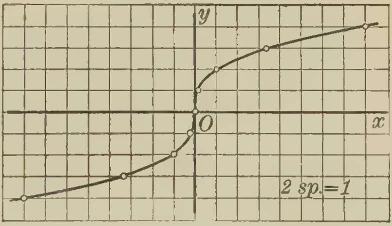


FIG. 12

Here, to avoid the extraction of cube roots, it will be convenient to write the equation in the form

$$x = \tfrac{1}{2}\,y^3$$

and assign values to y. Since x becomes very large as y increases, we will assign to y only small values, including some fractional ones:

y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
x	0	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{27}{16}$	4	$-\frac{1}{16}$	$-\frac{1}{2}$	$-\frac{27}{16}$	-4

.

As in the case of the straight line above, this curve evidently extends indefinitely in both directions; the portion containing the above points is shown in Fig. 12, with two spaces as the unit.

(*c*) Trace the curve

$$x^2 + y^2 = 1.$$

Writing the equation in the form

$$y = \pm \sqrt{1 - x^2},$$

we see that to every value of x there are two values of y; further, that y is real only when x is numerically less than 1.

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$-\frac{1}{3}$	$-\frac{2}{3}$	-1
y	± 1	$\pm \frac{2}{3}\sqrt{2}$	$\pm \frac{1}{3}\sqrt{5}$	0	$\pm \frac{2}{3}\sqrt{2}$	$\pm \frac{1}{3}\sqrt{5}$	0

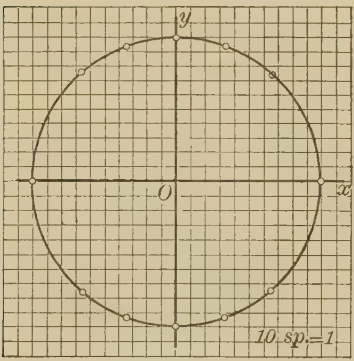


FIG. 13

To plot these points, we replace the irrational values of y by decimal approximations, correct to two places: $\frac{2}{3}\sqrt{2} = 0.94$, $\frac{1}{3}\sqrt{5} = 0.75$. The curve is shown in Fig. 13, with 10 spaces as the unit; it appears to be a circle of radius 1 with center at O. This is actually the case, since the equation evidently states that the distance from (x, y) to $(0, 0)$ is 1.

15. Scope of the term "curve." If the locus of every equation is to be spoken of as a curve, the meaning of that term must be extended so as to cover certain forms that would not in ordinary language be considered as curves. Thus the straight line is included as a special case; further, the locus of a single equation may consist of two or more disconnected branches (see, for instance, Fig. 16, p. 24), yet these several branches are said to constitute only a single curve.

Finally, we have as limiting cases curves made up of one or more separate points, and wholly imaginary curves: examples are furnished by the equations

$$x^2 + y^2 = 0,$$

$$x^2 + y^2 = -1,$$

which are satisfied only by the coördinates (0, 0), and by no real pairs of coördinates, respectively. As a rule, however, these limiting forms are of slight importance.

EXERCISES

Trace the following curves accurately, choosing a suitable scale in each case.

1. $y = 1 - 3x$.

2. $4x - 3y = 0$.

3. $2x - 3y = 6$.

4. $5x + 3y + 5 = 0$.

5. $y^2 + y = x$.

6. $y^2 + 3x = 0$.

7. $y = 1 + x - x^2$.

8. $y = x^3$.

9. $x^2 + y^2 = 25$.

10. $4x^2 + 4y^2 = 1$.

11. $x^2 + 4y^2 = 4$.

12. $9x^2 + y^2 = 36$.

13. $3y = x^3 - 3x^2 - 9x$.

14. $y = 3x - x^3$.

15. Of the above curves, which can be recognized from their equations as straight lines? Which as circles with center at O?

16. Symmetry. Two points P_1, P_2 are said to be placed *symmetrically with respect to an axis* λ, if λ is the perpendicular bisector of the segment P_1P_2. Each point is then said to be the image, or reflection, of the other in the line λ. A curve or other plane figure is said to be symmetric with respect to an axis if, corresponding to every point P_1 of the figure, the image-point P_2 in that axis also belongs to the figure. This means that the figure is unchanged by reflection in the axis — i.e. that the part of the figure on one side of the axis is the image of the part on the other side. Figure 14 shows a curve symmetric with respect to the indicated line.

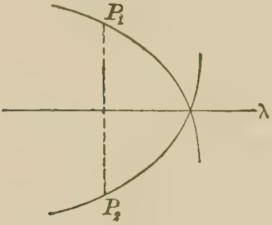


FIG. 14

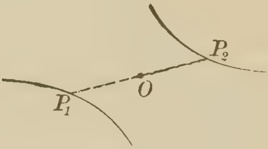


FIG. 15

Two points P_1, P_2 are said to be placed *symmetrically with respect to a point* O if O is the midpoint of P_1P_2. A plane figure is symmetric with respect to a point O if, corresponding to every point P_1 of the figure, there is a point P_2, also belonging to the figure, such that O is the midpoint of P_1P_2. The point O is called the *center of symmetry*, or simply the *center*. Figure 15 shows a curve having the point O as center.

17. Tests for symmetry. As direct consequences of the definitions of § 16 we deduce analytic tests for symmetry of a curve with respect to the coördinate axes and the origin.

THEOREM I: *A curve is symmetric with respect to the x-axis if its equation is unchanged* * *when y is replaced by* $-y$; *and conversely. Similarly for symmetry with respect to the y-axis.*

For, by the hypothesis, if any pair of coördinates x, y satisfy the equation, the coördinates $(x, -y)$ of the image-point with respect to the x-axis also satisfy the equation. The proof of the converse is left to the student.

THEOREM II: *A curve is symmetric with respect to the origin if its equation is unchanged when* x *is replaced by* $-x$ *and* y *by* $-y$ *simultaneously; and conversely.*

The proof is left as an exercise.

---

* More precisely, if the new form of the equation is *equivalent* to the original form — i.e. if each form is satisfied by all the values of the unknowns that satisfy the other.

When a curve is symmetric with respect to either axis or the origin, the process of tracing the curve may be much shortened. For instance, the curve

$$y^3 = 2x$$

occurring in example (*b*), § 14, is evidently symmetric with respect to O, since upon replacing x by $-x$ and y by $-y$ we get

$$(-y)^3 = -2x,$$

which is equivalent to the original equation. Knowing this, we need not assign any negative values to y; when the part of the curve lying in the first quadrant has been drawn, that in the third quadrant can be filled in by symmetry. Again, the curve

$$x^2 + y^2 = 1$$

(example (*c*), § 14) is symmetric to both axes, since x can be changed to $-x$ and y to $-y$ separately. Hence, when the curve has been drawn in the first quadrant all the rest can be filled in at once.

18. Intercepts on the axes. The points where a curve crosses the axes are often easily found, and if so they should be plotted as an important aid in tracing the curve. To find the points where the curve crosses Ox we must evidently put $y = 0$ and solve for x; to find the intersections with Oy we put $x = 0$ and solve for y. Of course in some cases the solution cannot be carried out on account of algebraic difficulties.

The distances cut off by a curve on the axes are called its *x- and y-intercepts.*

As already stated (§ 14), the locus of every equation of the first degree is a straight line. Hence, to draw the locus of such an equation it is only necessary to find the points of intersection with the axes and draw the straight line through them (unless the line passes through the origin or is parallel to an axis).

19. Curve tracing. In tracing a curve, the results of the preceding articles should be made use of whenever available. For concreteness, the process may be reduced to a definite schedule of steps, as follows:

(1) *Test the curve for symmetry with respect to each axis and the origin.*

(2) *Find the intersections with the axes.*

(3) *Determine a suitable number of additional points.*

Example: Trace the curve

$$x^2 - y^2 = 4.$$

(1) The curve is symmetric to both axes and the origin.

(2) When $y = 0$, $x = \pm 2$; when $x = 0$, y is imaginary, so that the curve does not intersect Oy.

(3) Writing the equation in the form

$$y = \pm \sqrt{x^2 - 4},$$

we see that y is imaginary if x is numerically less than 2. Assigning to x values greater than 2, we get the following points:

x	3	4	5	6
y	$\pm \sqrt{5}$	$\pm 2\sqrt{3}$	$\pm \sqrt{21}$	$\pm 4\sqrt{2}$

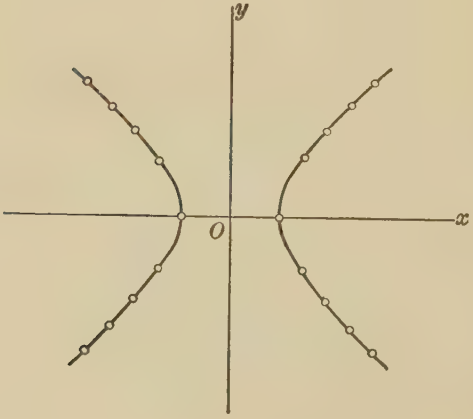


Fig. 16

On account of the symmetry with respect to Oy, the portion of the curve corresponding to negative values of x can be obtained by reflection.

The curve is shown in Fig. 16; it consists of two separate branches extending to infinity.

EXERCISES

Without formal proofs, state how many axes of symmetry are possessed by each of the following figures.

1. A circle.

2. (*a*) A straight line; (*b*) a straight line segment. *Ans.* (*b*) Two.

3. A triangle. Discuss special cases fully.

4. A quadrilateral. Discuss all special cases.

5. A regular hexagon.

6. Which of the figures in Exs. 1–5 possess centers?

7. Show that two circles taken together always have one axis of symmetry. When are there two such axes? When more than two?

8. Under what circumstances do two circles have a center of symmetry?

9. When do three circles have an axis of symmetry? When more than one?

10. When do three circles have a center of symmetry?

11. State and prove the converse of Theorem I, § 17.

12. Prove Theorem II, § 17.

13. State and prove the converse of Theorem II, § 17.

14. Test each of the curves of Exs. 5–14, p. 21, for symmetry with respect to each axis and the origin.

15. Prove analytically that if a curve is symmetric with respect to each of two perpendicular lines, it is symmetric with respect to their point of intersection. Show by means of an example that the converse is not true.

16. Prove the theorem of Ex. 15 geometrically.

17. Obtain a test for symmetry with respect to the line through O bisecting the first and third quadrants.

18. Obtain a test for symmetry with respect to the line through O bisecting the second and fourth quadrants.

19. Find the intercepts on the axes of the straight lines in Exs. 1–4, p. 21.

20. Find the intercepts on the axes of the curves in Exs. 5–14, p. 21.

Draw the following straight lines.

21. $y - x = 2.$

22. $3x + y = 6.$

23. $5x - 6y = 0.$

24. $20x - 15y = 1.$

25. $2x + 3y + 30 = 0.$

26. $x = 2.$

Trace the following curves accurately, on a suitable scale, applying each of the steps of § 19.

27. $3y = 2x^2.$

28. $x^2 - 3x + y - 4 = 0.$

29. $4x^2 + y^2 = 9.$

30. $y^2 = x - y.$

31. $x^2 - y^2 + 9 = 0.$

32. $x^2 + y^2 = 4x.$

33. $4x^2 - y^2 = 4.$

34. $50x^2 + 100y^2 = 1.$

35. $y = x(x+1)(x-3).$

36. $xy = 1.$

37. $xy + 10 = 0.$

38. $y = \dfrac{x}{x^2+1}.$

39. $y = \dfrac{x^2}{x^2+1}.$

40. $y = \frac{1}{3}x^3 - 4x.$

41. $xy^2 = 1.$

42. $y = \frac{1}{4}x^4 - 2x^2.$

20. Consequences of the definition of locus. From the definition of the locus of an equation we may deduce certain rules which are so important that, in spite of their obviousness, it seems worth while to state them explicitly.

By virtue of the definition, a point lies on a curve *if and only if its coördinates satisfy the equation of the curve.* Hence:

RULE I: *To determine whether a given point lies on a given curve, substitute its coördinates for x and y in the equation of the curve.*

Example: (*a*) The point (2, 12) lies on the curve

$$y = 3x^2$$

because

$$12 = 3 \cdot 4;$$

the point (− 1, − 3) does not lie on the curve because

$$-3 \neq 3 \cdot 1.$$

RULE II: *To express analytically the condition that a point shall lie on a curve, write the equation of the curve with the coördinates of the point substituted for x and y.*

Examples: (*b*) The point (x_1, y_1) lies on the curve

$$x^2 + y^2 = a^2$$

if and only if

$$x_1^2 + y_1^2 = a^2.$$

(*c*) Determine the constant a so that the point $(-1, -3)$ shall lie on the curve

$$y = ax^2.$$

By merely expressing analytically the data of the problem we get

$$-3 = a \cdot 1,$$

whence

$$a = -3$$

and the equation of the curve is

$$y = -3x^2.$$

Check: $-3 = -3 \cdot 1.$

RULE III: *To find the ordinate of a point on a curve, being given its abscissa, substitute the given abscissa for x in the equation of the curve and solve for y.* Similarly we may find the abscissa when the ordinate is given.

21. Factorable equations. In algebra, when the product of two or more factors is equated to 0, the equation thus formed is satisfied whenever any one of the factors is 0. For instance, the equation

(1) $3(x+2y)(x^2-y^2)=0$

is satisfied whenever

$$x + 2y = 0,$$

$$x + y = 0,$$

or $x - y = 0.$

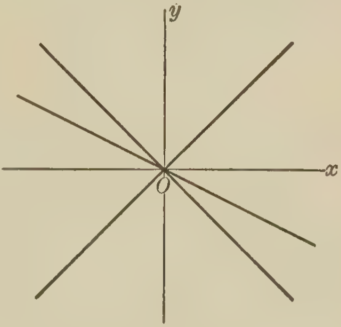


FIG. 17

Hence, the locus of an equation whose right member is 0 and whose left member can be broken up into factors consists of all points whose coördinates when substituted in the equation cause any one of the factors to vanish. That is, the curve represented by such an equation breaks up into several distinct curves, viz. the loci of the equations formed by equating the several factors separately to 0. Thus equation (1) above represents the three curves (straight lines)

$$x + 2y = 0,\ x + y = 0,\ x - y = 0.$$

22. Algebraic transformation of equations. When an algebraic operation of any kind is performed upon an equation, the locus is evidently unchanged provided that, after the transformation, the equation is satisfied by all those pairs of values of x and y, and only those, which satisfied it originally. This is clearly the case with such operations as transposition of terms from one member to the other, multiplication of both members by a constant factor, etc. On the other hand, the locus is in general changed by introducing or striking out a variable factor, squaring both members, etc. For instance, the equation

$$y = x$$

represents a certain straight line; the equation

$$y^2 = x^2$$

holds not only when $y = x$, but also when $y = -x$, and thus represents two straight lines.

EXERCISES

Determine whether the given points lie on the given curve.

1. Curve $2x + 3y = 5$; points $(1, 1)$, $(5, -5)$, $(0, \frac{5}{3})$, $(13, 6)$.

2. Curve $y = x^2 - 1$; points $(-2, 3)$, $(15, 4)$, $(2, -3)$, $(-\frac{1}{2}, -\frac{5}{4})$.

3. Curve $x^2 + y^2 = 25$; points $(-5, 0)$, $(\pm 3, \pm 4)$, $(2\sqrt{5}, \sqrt{5})$, $(2\sqrt{3}, 3\sqrt{2})$.

4. Curve $y^2 = 4ax$; points $(2a, a)$, $(-a, 2a)$, $(4a, -4a)$, $(\frac{1}{4}a, a)$.

5. Curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; points (a, b), $(-a, 0)$, $\left(\sqrt{a^2 - b^2}, \frac{b^2}{a}\right)$.

6. Determine k so that the straight line $x + 2y = k$ shall pass (*a*) through $(4, 1)$; (*b*) through $(-6, -5)$; (*c*) through $(0, 0)$.

7. For what value of a does the curve $y = ax^3$ pass through $(-1, -3)$? Through $(4, 2)$? Through $(0, 0)$?

8. Determine m so that the line $y = mx + 2$ shall pass (*a*) through $(1, 3)$; (*b*) through $(2, -2)$; (*c*) through $(0, 2)$. Can this line be made to pass through $(0, 0)$?

9. What is the condition that the curve $y = ax^2 + bx + c$ shall pass through $(0, 0)$? Through $(2, 1)$? Through $(-1, c)$?

10. Given that the point (x_1, y_1) lies on the curve $y^2 = 4\,ax$, express this fact analytically.

11. Under what circumstances does the point (h, k) lie on the line $3\,x + 4\,y = 6$?

12. In the curve $y = 6\,x^2$, find (*a*) the point whose abscissa is 2; (*b*) the points whose ordinate is 3.

13. In the curve $y^2 - x - 3y + 2 = 0$, find (*a*) the point whose ordinate is 3; (*b*) the points whose abscissa is 0; (*c*) the points whose abscissa is 3; (*d*) the points whose abscissa is -4. Interpret the result (*d*) geometrically.

14. Verify that the equation of the straight line joining the points $(3, 2)$, $(-2, 5)$ is $3\,x + 5\,y = 19$.

Trace the following curves

15. $x^2 - 3\,xy = 0$.

16. $x^3y = 4\,xy^3$.

17. $x^2 - 4\,xy + 4\,y^2 = 0$.

8. $x^2 - 4\,xy + 4\,y^2 = 1$.

19. $(x^2 + y^2 - 1)\,(y^3 - 2\,x) = 0$.

20. $xy - 3\,y^2 - 2\,x + 6\,y = 0$.

21. Given an equation in which x is a factor of every term, how is the locus affected if this factor be stricken out?

23. Points of intersection of two curves. The points of intersection of two curves are points whose coördinates *satisfy both equations*, and there are evidently no other points having this property. Hence *the points of intersection of two curves are found by solving the equations of the curves as simultaneous equations.*

After solving two simultaneous equations the results should always be tested by substituting the values of x and y in both equations and noting whether the equations hold.

Example: Find the points of intersection of the straight line

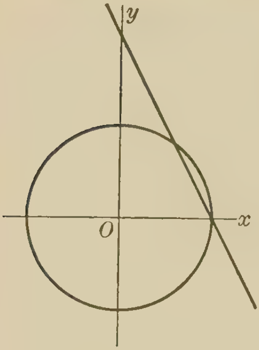


FIG. 18

$$(1) \qquad 2x + y = 10$$

and the circle

$$(2) \qquad x^2 + y^2 = 25.$$

Substituting the value of y from (1) in (2), we get

$$x^2 + (10 - 2x)^2 = 25,$$

or

$$5x^2 - 40x + 75 = 0,$$

$$x^2 - 8x + 15 = 0,$$

whence

$$x = 3 \text{ or } 5.$$

By (1),

$$y = 4 \text{ or } 0,$$

and the points are (3, 4), (5, 0).

Check:

$$6 + 4 = 10,$$

$$9 + 16 = 25;$$

also

$$10 + 0 = 10,$$

$$25 + 0 = 25.$$

The curves are shown in the figure.

24. Importance of checks. It may be worth while to insert at this point a remark on the subject of checks. This is a matter to which the average student pays far too little attention. The habit of verifying results is a most valuable one in any line of investigation, and there is no place where that habit can be more easily formed than in the study of mathematics. In analytic geometry, particularly, there is almost always a simple and obvious way of proving results, sometimes with only reasonable certainty, but usually with absolute certainty. As the course proceeds the student is urged to be on the lookout for methods of checking, and to apply those methods habitually.

25. Non-intersecting curves. When the values of x or y found by solving two simultaneous equations are imaginary, the curves intersect in imaginary points, which means of course that in the ordinary sense they do not intersect at all. As an illustration the student may show that the curves

$$2x + y = 15,$$
$$x^2 + y^2 = 25$$

intersect in the points $(6 + 2i, 3 - 4i)$ and $(6 - 2i, 3 + 4i)$, where $i = \sqrt{-1}$; the two curves are shown in Fig. 19.

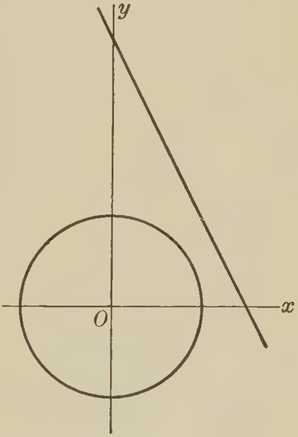


Fig. 19

It may happen that two equations cannot exist simultaneously for any values of x and y, even imaginary ones, in which case they are said to be *incompatible*. The corresponding curves not only do not intersect, but they do not even have imaginary points in common.* An illustration is furnished by the equations

$$x^2 + y^2 = 25,$$
$$x^2 + y^2 = 9.$$

For, if we subtract the second equation from the first, we obtain the absurdity

$$0 = 16.$$

These curves are concentric circles, as shown in Fig. 20.

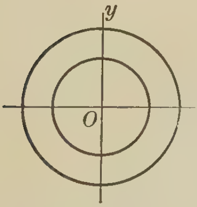


Fig. 20

26. Number of points of intersection. In algebra, the *degree* of a term involving x and y is the sum of the exponents of x and y in the term. Thus the terms x^2, $3xy$, $2y^2$ are of the second degree; $3x^4$, $2xy^3$ are of the fourth degree; $\frac{2y}{3x^3}$ is of degree -2.

---

* In § 125, even this limiting case will be given an analytic interpretation.

The *degree* of an algebraic equation in x and y is that of the term of highest degree when the equation is rationalized and cleared of fractions.* Thus the equation

$$\sqrt{3}\,x + \tfrac{2}{3}\,y = 5 \tag{1}$$

is of the first degree; the equation

$$y = \frac{1}{x^2}$$

is of the third degree; the equation

$$y = x^{2/3}$$

is of the third degree.

A curve whose equation is of the nth degree is spoken of briefly as a *curve of the nth degree.*

It is shown in algebra that when two equations in x and y are solved as simultaneous, the number of solutions (i.e. the number of pairs of values of x and y satisfying both equations) is not greater than the product of the degrees of the equations.† Hence *the number of points of intersection of two curves is not greater than the product of the degrees of their equations.*

EXERCISES

Find the points of intersection of the following curves; check the results.

1. $2x - 23y + 6 = 0, \quad x - 6y = 8.$

2. $7x - 8y = 1, \quad 3x + 5y + 5 = 0.$

3. $4x + 11y = 0, \quad 2x + 6y = 3.$

4. $3x - 3y - 8 = 0, \quad y = x - 2.$

5. $x^2 + y^2 = 5, \quad 4x + y = 7.$

6. $x^2 + y^2 = 6, \quad x^2 + y^2 - 4x + 2 = 0.$

---

* As regards the variables only: irrational and fractional constants may be present, as in (1).

† The number of solutions is in general just equal to the product of the degrees, but there are many exceptional cases.

7. $y = 3x^2 + 5x + 7, \quad 4x + y = 1.$

8. $x^2 + y^2 = 10, \quad 2x + 3y = 9.$

9. $x^2 + y^2 + 2x = 9, \quad 3x - y = 7.$

10. $2x + 5y^2 = 44, \quad y = 1 - 2x.$

11. $y = 4x^2 - 16x, \quad y = x^2 - 4x - 15.$

12. $y^3 = 2x, \quad y = 4x.$

13. $x^2 + y^2 = 4y, \quad 3x^2 + y^2 = 30.$

Determine the degree of each of the following equations.

14. $3x - 2xy = 6.$

15. $x^3 - y^3 = 3xy.$

16. $\sqrt{2}(x - 2) = 3y.$

17. $y = \pm\sqrt{x}.$

18. $y = \dfrac{1}{x+1}.$

19. $y = \dfrac{x}{x^2+1}.$

20. Show that the equation $x^{1/2} \pm y^{1/2} = \pm a^{1/2}$ is of the second degree.

21. Show that the equation $x^{2/3} + y^{2/3} = a^{2/3}$ is of the sixth degree.

22. Prove that if three lines form an equilateral triangle, the equation of at least one of them must contain irrational coefficients. (Cf. Ex. 23, p. 13.)

CHAPTER III

LOCI

27. Path of a moving point. Hitherto we have thought of a curve as composed of the aggregate of points whose coördinates satisfy a certain equation. A somewhat different conception of a curve is obtained by thinking of it as the *path*, or *locus, of a point which moves according to a certain law*. Thus the straight line which bisects the first and third quadrants may be thought of as composed of all points whose coördinates satisfy the equation

$$y = x,$$

or it may be considered as the path of a point which moves so as to remain always equidistant from the axes. The latter conception of a curve is of great importance in analytic geometry.

If the equation of a curve is not given it must be found before the curve can be studied analytically. When the curve is defined as the path of a moving point, the statement of the law of motion usually suggests an equality between certain distances or other quantities involving the coördinates of the moving point. We always denote the coördinates of the moving point by (x, y), and try to obtain (by the formulas of analytic geometry) expressions for the distances or other quantities involved in the statement of the law of motion, after which it is usually a simple matter to write out the equation which is suggested by the law.

Owing to the fact that we have as yet developed only a small body of analytic theory, the number of "locus problems" that we can solve at present is rather limited, but the general method of attack is sufficiently well brought out by the following simple examples and exercises.

Examples. (*a*) A point moves so as to remain always equidistant from the points P_1: (3, 2) and P_2: (− 1, 5). Find the equation of its locus.

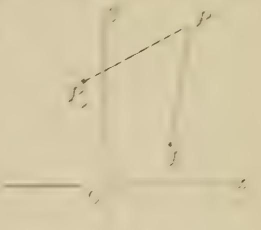


FIG. 21

The law of motion states that the distance from the moving point P: (x, y) to the point (3, 2) equals the distance from the point (x, y) to the point (− 1, 5). Hence, by the formula for the distance between two points,

$$\sqrt{(x-3)^2+(y-2)^2}=\sqrt{(x+1)^2+(y-5)^2}.$$

This equation expresses the given law of motion and is therefore the equation of the locus; it remains only to simplify the equation. Squaring and expanding, we get

$$x^2-6x+9+y^2-4y+4=x^2+2x+1+y^2-10y+25,$$

and this reduces to

$$8x-6y+13=0$$

which shows that the locus is a straight line. It is of course the perpendicular bisector of the line joining the fixed points.

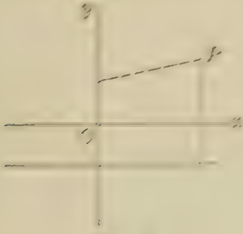


FIG. 22

(*b*) A point moves so as to remain equidistant from the line $y=-1$ and the point (0, 1). Find the equation of its locus.

The distance of any point P: (x, y) from the line $y=-1$ is $y+1$, as is evident at once from the figure. Hence we have

$$y+1=\sqrt{x^2+(y-1)^2},$$

$$y^2+2y+1=x^2+y^2-2y+1,$$

$$x^2=4y.$$

To prove exactly that a curve and its equation as found in the present chapter correspond to each other in the manner required by the definition of "locus of an equation" (§ 14), it is clearly necessary to show (*a*) that the coordinates of every point of the locus satisfy the equation; (*b*) that every point whose coördinates satisfy the equation lies on the locus. The method outlined in the second paragraph of this article, and applied in the above examples, establishes only the first of these statements. However, in most cases the second part of the proof can be carried out by merely reversing the steps already taken, and therefore will usually be omitted.*

EXERCISES

1. Draw the straight line in example (*a*) above by finding its intercepts, and verify by measurement that its points are equidistant from the given points.

2. Trace the curve in example (*b*) from its equation, and verify by measurement that its points are equidistant from the given line and the given point.

In each of the following cases, find the equation of the locus, and draw the curve.

3. A moving point is always equidistant from the points (0, 0) and (4, 0).

4. A moving point is always equidistant from the points (4, 0) and (– 6, – 3).

5. A point moves so that its distance from the point (0, 0) is always twice its distance from the point (0, – 6).

Ans. $x^2 + y^2 + 16y + 48 = 0$.

---

* The second part of the proof in example (*b*), for instance, is as follows. Given the equation

$$4y = x^2, \tag{1}$$

let us add $y^2 - 2y + 1$ to each member, thus obtaining

$$y^2 + 2y + 1 = x^2 + (y-1)^2,$$

whence we have either

$$y + 1 = \sqrt{x^2 + (y-1)^2}, \tag{2}$$

or

$$y + 1 = -\sqrt{x^2 + (y-1)^2}. \tag{3}$$

But by the conditions of the problem the quantity $y + 1$ is necessarily positive; hence (3) must be rejected. Equation (2) therefore holds whenever (1) holds. But (2) states that the point (x, y) is equidistant from the line $y = -1$ and the point (0, 1); thus that point belongs to the locus.

6. A point moves so that its distance from the point (3, 2) is always one third of its distance from (5, 2).

7. A point moves so that its distance from the x-axis is three times its distance from the y-axis.

8. A moving point is always at the distance 3 from the origin.

9. A moving point is always at the distance 6 from the point (2, 3).

10. A moving point is always at the distance a from the point $(a, 0)$.

11. A moving point remains always equidistant from the line $x = 3$ and the point $(-3, 0)$. *Ans.* $y^2 + 12x = 0$.

12. A moving point is always equidistant from the point (4, 3) and the line $y = 6$.

13. A point moves so that its distance from the point (1, 0) is always one half of its distance from the line $x = 4$.

14. A point moves so that its distance from the point (0, 5) is always two thirds of its distance from the x-axis.

15. A point moves so that its distance from the point (3, 2) is always twice its distance from the y-axis.

16. The area of the triangle whose vertices are the moving point, the origin, and the point (1, 0) is equal to 3. *Ans.* $y = 6$, or $y = -6$.

17. A point moves so that the sum of its distances from the points (4, 0) and $(-4, 0)$ is 10. *Ans.* $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

18. A point moves so that the sum of its distances from the points (0, 0) and (0, 2) is 3.

19. A point moves so that the difference of its distances from (4, 3) and (8, 3) is 1.

20. A point forms with the points (3, 2), $(-1, 5)$ a triangle of area 2; find the equation of its locus.
Ans. $3x + 4y = 13$, or $3x + 4y = 21$.

21. Two vertices of a triangle are (1, 1), (3, 6); the area is 3. Find the locus of the third vertex.

22. The base of a triangle is the line joining (4, 3), (0, 6); the altitude is 2. Find the locus of the third vertex.

23. If the triangle of Ex. 22 is isosceles, find the third vertex.

24. The base of a triangle joins the points $(3, 5)$, $(-1, 6)$; the altitude is 3. Where does the third vertex lie?

25. The hypotenuse of an isosceles right triangle joins the points $(3, 5)$, $(2, 3)$. Find the third vertex. *Ans.* $(\frac{7}{2}, \frac{7}{2})$, or $(\frac{3}{2}, \frac{9}{2})$.

26. Two vertices of a right triangle are $(0, 2)$, $(4, 8)$, with the right angle at $(4, 8)$. Find the locus of the third vertex.

27. In Ex. 26, if the area of the triangle is 4, find the third vertex.

28. The hypotenuse of an isosceles right triangle joins the points $(1, 1)$, $(-3, 3)$. Find the third vertex.

29. One diagonal of a square joins the points $(3, -4)$, $(2, 3)$. Find the other vertices. *Ans.* $(6, 0)$, $(-1, -1)$.

30. A chord of a circle of radius $\sqrt{29}$ joins the points $(7, 2)$, $(1, -6)$. Find the center of the circle.

28. Loci defined geometrically. Another important method of defining a curve is by means of purely geometric data. Thus we may have a curve actually drawn, and be required to find its equation from the known geometric properties of the figure, as in example (*a*) below; or a geometric construction may be given by means of which the points of the curve are determined, as in (*b*) and (*c*).

In such cases we *assume a point of the curve in a general position*, and denote its coördinates by (x, y). Then the problem is merely to express some characteristic property of the curve by means of an equation involving x, y, and the constants of the problem. (By "characteristic property" is meant, of course, one that holds for all the points of the curve, and for no other points.) No matter what property is used, the result must be the equation of the curve.

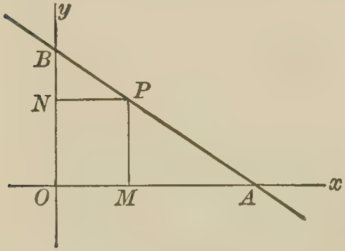


FIG. 23

Examples: (*a*) Find the equation of the straight line whose intercepts on the axes are $OA = 3$ and $OB = 2$ respectively.

Assume a point P: (x, y) in a general position on the line. It is clear that the triangles MAP and OAB are similar if and only if P is on the line. Hence if the fact that these triangles are similar be expressed by an equation involving x and y, that equation must represent the line. Now

$$\frac{MP}{OB} = \frac{MA}{OA};$$

that is,

$$\frac{y}{2} = \frac{3 - x}{3}.$$

This is the equation of the line. Clearing of fractions and transposing, we get the simpler form

$$2x + 3y = 6.$$

Check: When $y = 0$, $x = 3$; when $x = 0$, $y = 2$.

(*b*) Find the locus of the centers of circles passing through the point P_1: $(0, 1)$ and tangent to the line $y + 1 = 0$.

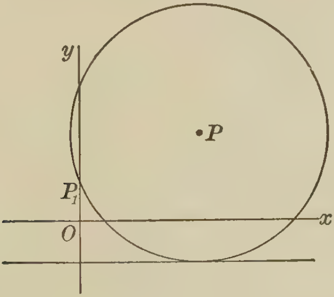


FIG. 24

Let us draw one * of the circles in question, and call its center the point P: (x, y). It appears at once that P is equidistant from the line $y + 1 = 0$ and the point $(0, 1)$, so that we have here the same locus as in example (*b*), § 27:

$$y + 1 = \sqrt{x^2 + (y - 1)^2}, \text{ etc.}$$

---

* To draw more than one would merely obscure the figure, without helping us to find the equation of the locus.

(*c*) A circle of radius 2 is described about the origin; a circle of radius 1 rolls on the outside of this fixed circle. Find the locus of the center of the rolling circle.

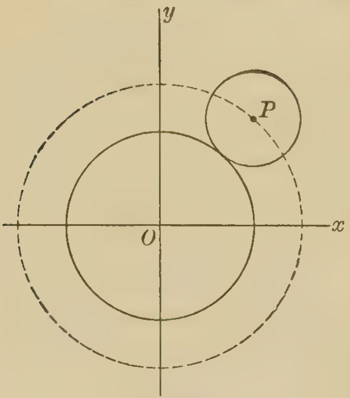


FIG. 25

The distance of the point $P: (x, y)$ from the origin is 3; therefore the required equation is

$$\sqrt{x^2 + y^2} = 3,$$

or

$$x^2 + y^2 = 9.$$

This is of course a circle of radius 3 about the origin (the dotted circle of Fig. 25).

EXERCISES

1. Solve example (*a*) by making use of the fact that the triangles MAP and NPB are similar.

2. Solve example (*c*) if the smaller circle rolls on the inside of the larger one.

3. Find the locus of the centers of circles tangent to both cöordinate axes.

4. Find the equation of the perpendicular bisector of the line joining (3, 3) and (5, – 7). *Ans.* $x - 5y = 14$.

5. Find the locus of the centers of circles through the points (2, 3), (5, – 2).

6. Find the locus of the centers of circles through the points (4, 0), (– 2, – 4).

7. The base of an isosceles triangle is the line from (2, 4) to (3, – 5); find the locus of the vertex.

8. The hypotenuse of a right triangle is the line from (0, 0) to (0, 8); find the locus of the third vertex.

9. The hypotenuse of a right triangle is the line from (3, – 7) to (2, 6); find the locus of the third vertex.

10. Find the equation of the straight line through the origin and the point (6, 4).

11. Find the line through the origin and the point $(-7, 3)$.

12. Find the equation of the line whose x- and y-intercepts are respectively $\frac{2}{3}$ and $\frac{7}{5}$.

13. Find the equation of the line whose intercepts are -5 and 2.

14. Find the equation of the line whose intercepts are -4 and -6.

15. Find the equation of a circle of radius 3 with center at $(2, 5)$.

16. One of the equal sides of an isosceles triangle is the line from $(4, 2)$ to $(3, 3)$. Find the locus of the third vertex.

17. Find the equation of a circle with center at $(\frac{1}{3}, -\frac{5}{3})$ and passing through $(-1, -\frac{1}{3})$.

18. Find the equation of a straight line through the point $(2, 6)$ making an angle of 45° with Ox.

19. A point moves so that the lines joining it to the points $(a, 0)$ and $(-a, 0)$ are perpendicular Find the equation of its locus.

20. A variable circle is tangent to two fixed circles of radius 1 with centers at $(5, 0)$ and $(1, 7)$. Find the locus of the center of the variable circle.

21. A circle passes through the point $(2, 3)$ and touches the line $y = 6$. Find the locus of its center.

22. A circle passes through the point $(\frac{1}{2}, -\frac{3}{4})$ and touches the line $2x + 1 = 0$. Find the locus of its center.

23. A moving circle is tangent to the y-axis and to a circle of radius 1 with center at $(2, 0)$. Find the locus of the center of the moving circle.

24. Find the equation of a straight line parallel to and at a distance 4 from the line joining $(3, 5)$ and $(-1, 2)$.

Ans. $3x - 4y - 9 = 0$, $3x - 4y + 31 = 0$.

25. A circle of radius $\sqrt{5}$ rolls on the straight line drawn through the points $(2, 0)$ and $(0, 4)$. Find the locus of its center.

CHAPTER IV

THE STRAIGHT LINE

29. Introduction. Stated broadly, the purpose of plane analytic geometry, as already remarked, is to investigate the form and properties of plane curves. The first problem to be solved in beginning the study of a curve is to obtain the equation of the curve from its definition; this problem has been studied in the preceding chapter. Unless the curve is actually drawn to begin with, which is not usually the case, we have next to trace the curve from its equation; this we learned how to do in Chapter II. Finally, we obtain the properties of the curve, investigate its relations with other curves, etc., by the methods of analysis; what these methods are and how they are applied will appear as we proceed.

To study to advantage curves whose equations are of higher than the second degree — the so-called "higher plane curves" — the methods of differential calculus are required. Further, the straight line (curve of the first degree), and the "conic sections," as the curves of the second degree are called, are of especial importance in the applications. For these reasons, we shall study in detail only these two classes of curves, taking up the straight line in the present chapter.

30. Line parallel to a coördinate axis. If a straight line is parallel to the y-axis, its equation is evidently

$$x = k,$$

where k is the distance of the line from the axis; and conversely. For, all points on that line, and no other points, have the abscissa k.

Similarly, a line parallel to and at a distance k from the x-axis has the equation

$$y = k.$$

Thus we have the

Theorem: *An equation of the first degree in one variable represents a straight line parallel to the axis of the other variable; and conversely.*

Fig. 26 shows the lines $x = \frac{2}{3}$, $y = -1$.

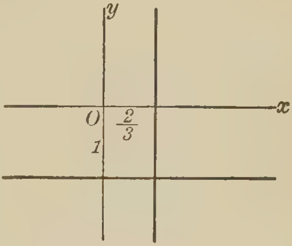


Fig. 26

31. Line through a given point in a given direction: point-slope form. Let us try to find the equation of a straight line passing through a given point P_1: (x_1, y_1) in a given direction, — i.e. having a given slope $m = \tan\alpha$. Following the method of § 28, we assume a point P: (x, y) on the line in a general position, and try to express some property of the figure by means of an equation involving x, y, and the constants of the problem. Now in the triangle P_1MP,

$$\tan\alpha = \frac{MP}{P_1M} = m;$$

that is,

$$\frac{y - y_1}{x - x_1} = m,$$

or

$$(1) \qquad y - y_1 = m(x - x_1).$$

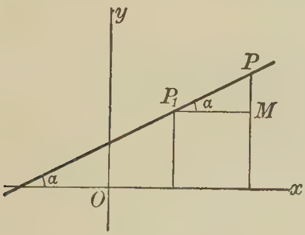


Fig. 27

This is called the *point-slope form* of the equation of the straight line. It is one of several so-called *standard forms* of the equation of the straight line which will be developed in this chapter.

The point-slope form fails in case the line is parallel to the y-axis, since for such a line $m = \tan 90°$, which is non-existent. But by § 30, the equation of a line through (x_1, y_1) parallel to Oy is simply

$$x = x_1.$$

When the slope and one point of a line are given, the line can evidently be drawn, if desired, without writing the equation. Thus, to draw the line of slope $\frac{3}{2}$ through the point (4, 2): starting at (4, 2), we measure off 2 units to the right and then 3 upward (or 4 to the right and 6 upward, or 2 to the left and 3 downward, etc.); through the point thus reached and (4, 2) we draw the line.

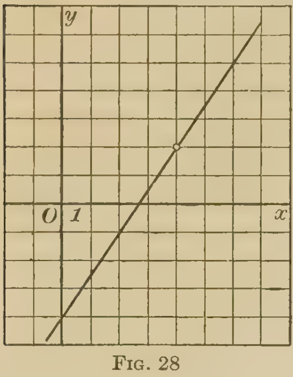


FIG. 28

EXERCISES

1. Draw the lines (*a*) $x = 0$, (*b*) $y = 0$, (*c*) $x + 4 = 0$, (*d*) $3x = 5$, (*e*) $2y - 1 = 0$, (*f*) $3 + 4y = 0$.

2. What is represented by the equations (*a*) $x^2 - 3x = 4$, (*b*) $4y^2 + 12y + 9 = 0$, (*c*) $y^2 + 2y + 2 = 0$?

3. Without writing the equations, draw the following lines:

(*a*) of slope 4 through (0, 0);
(*b*) of slope $-\frac{2}{3}$ through (0, 0);
(*c*) of slope -3 through (2, 1);
(*d*) of slope $\frac{3}{5}$ through (3, -2);
(*e*) of slope $-\frac{5}{2}$ through (-6, -2);
(*f*) of slope $\frac{5}{6}$ through (-7, 3).

Write the equations of the following lines, and draw the figures.

4. Through (4, -2) (*a*) parallel to the y-axis; (*b*) parallel to the x-axis.

5. Through (1, 6) (*a*) parallel to the line $2x + 5 = 0$; (*b*) perpendicular to the same line.

6. Through (-2, 5) (*a*) parallel to the line $5 - 3y = 0$; (*b*) perpendicular to the same line.

7. Through (-3, 2) with slope $\frac{1}{3}$.

8. Through (2, 0) with slope -5.

9. Through (1, 1) with slope $-\frac{3}{5}$.

10. Through (1, 5) making an angle of 60° (*a*) with the x-axis; (*b*) with the y-axis.

11. Through the origin with slope m.

12. Having the slope 5 and y-intercept 3.

13. Having the slope $-\frac{2}{3}$ and x-intercept -2.

14. Through $(5, -1)$ perpendicular to a line of slope 4.

15. Through $(2, -2)$ parallel to the line joining $(2, -1)$ and $(3, 4)$.

16. Through $(0, 1)$ parallel to the line through $(\frac{3}{5}, -\frac{5}{2})$ and $(4, 1)$.

17. Through the origin parallel to the line whose intercepts on the axes are a and b respectively.

18. Through $(2, 6)$ (a) parallel, (b) perpendicular, to the line through $(4, 3)$ and $(-1, 3)$.

19. Through the origin (a) parallel, (b) perpendicular, to the line through $(2, -3)$ and $(2, 1)$.

20. Through $(1, 4)$ perpendicular to the line through $(\frac{2}{3}, -\frac{1}{6})$ and $(-\frac{3}{4}, -\frac{2}{3})$.

21. Through $(-\frac{2}{5}, -\frac{6}{7})$ perpendicular to the line from $(0, 0)$ to $(5, -2)$.

22. Trace the curve $x^3 = x$.

23. Trace the curve $y^3 - 6y^2 + 9y = 0$.

24. By a new method, find the equation of the perpendicular bisector of the line joining the points $(3, 3)$, $(5, -7)$. (Ex. 4, p. 40.)

25. Find in two ways the equation of the perpendicular bisector of the line from $(5, 2)$ to $(-4, -3)$.

26. Find in two ways the locus of points equidistant from $(0, 1)$ and $(3, 0)$.

27. Two circles of radius 10 are drawn with centers at $(7, 1)$ and $(-7, 3)$. Find the equation of their common chord, and the length of the chord.

28. Does the line of slope 2 through $(5, 1)$ also pass through $(20, 32)$? Through $(-2, -13)$?

29. Three vertices of a rectangle are $(3, -2)$, $(7, 6)$, $(3, 8)$. Find the fourth vertex by a new method. (Ex. 7, p. 11.)

30. One side of a square joins the points $(5, -3)$, $(3, -2)$. Find the other vertices.

31. Prove that the altitudes of any triangle are concurrent.

32. Line through two points. By means of the point-slope form the equation of the straight line through two given points may be written down at once (unless the two points have the same abscissa, in which case the line is parallel to Oy). For we can find the slope of the line by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and can then use this value of m and the coördinates of either of the given points in equation (**1**), § 31.

When the answer has been reduced to the final form, it should always be checked by substituting the coördinates of each of the given points.

Example: Find the equation of the line through the points $(3, -5)$, $(-6, -3)$.

In this case

$$m = \frac{-3+5}{-6-3} = -\tfrac{2}{9},$$

whence the equation of the line is

$$\begin{aligned} y + 3 &= -\tfrac{2}{9}(x + 6), \\ 9y + 27 &= -2x - 12, \\ 2x + 9y + 39 &= 0. \end{aligned}$$

Check:

$$\begin{aligned} 6 - 45 + 39 &= 0, \\ -12 - 27 + 39 &= 0. \end{aligned}$$

33. Slope form. A very important special case of the problem of § 31 is that in which the given point is the point where the line crosses the y-axis, so that the line is determined by its slope and its y-intercept.

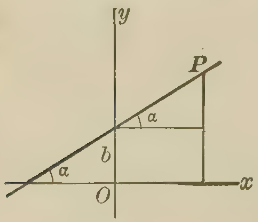


FIG. 29

Given a line of slope m whose y-intercept is b, let us assume a point P: (x, y) on the line. Then

$$\tan\alpha = \frac{y-b}{x} = m,$$

whence

$$(2) \qquad \boldsymbol{y = mx + b}.$$

This is known as the *slope form* of the equation of the straight line. It appears from this result that when the equation of a straight line is in the slope form — i.e. when it is *solved for* y — the coefficient of x is the slope and the constant term is the y-intercept.

EXERCISES

Find the equations of the following lines.

1. Through the points $(4, 3)$, $(6, 2)$.

2. Through the points $(-1, 0)$, $(-3, -4)$.

3. Through the points $(-2, -7)$, $(-\frac{3}{2}, \frac{2}{3})$.

4. Through the points $(\frac{7}{8}, \frac{5}{8})$, $(\frac{1}{3}, -\frac{3}{4})$.

5. Through the points $(1.2, -0.3)$, $(0, -1.75)$.

6. Through the points $(3, 5)$, $(-1, 5)$.

7. Through the points $(2, 0)$, $(2, -7)$.

8. Through the origin and the point of intersection of the lines $3x + 2y + 7 = 0$, $x - 5y = 8$.

9. The vertices of a triangle are the points $(3, 1)$, $(2, 5)$, $(-1, 7)$. Find the point of intersection of two of the medians, and verify that the third median passes through this point.

10. Three vertices of a parallelogram are $(2, 0)$, $(1, 3)$, $(3, 4)$. Find the fourth vertex.

11. Show that the equation of the straight line through two points may be written in the form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

12. Derive the formula of Ex. 11 directly, by the method of § 28.

13. Write the equation of the line having the y-intercept -3 and making an angle of 135° with Ox.

14. Write the equation of a line of slope $\frac{1}{2}$ and (*a*) passing through the origin; (*b*) having a y-intercept -2.

15. Draw the following lines by making use of the y-intercept and the slope:

(*a*) $y = \frac{2}{5}x - 2$; (*b*) $y = -3x + 3$;
(*c*) $y = -\frac{3}{4}x + 60$; (*d*) $y = 0.1x - 0.02$.

16. Derive the slope form as a special case of the point-slope form.

17. Write the equation of the straight line through $(2, -1)$ and (*a*) parallel, (*b*) perpendicular to $y = -\frac{7}{2}x + 1$.

18. Write the equation of the straight line (*a*) through $(5, 0)$ parallel to $y = -3x$; (*b*) through $(-2, -\frac{3}{5})$ parallel to $y = -\frac{1}{4}x + 2$. Draw the lines.

19. Write the equation of the straight line (*a*) through $(6, 2)$ perpendicular to $y = 2x + 4$; (*b*) through $(\frac{1}{10}, -\frac{3}{20})$ perpendicular to $y = -\frac{1}{5}x + 1$.

20. Two vertices of a right triangle are $(5, -3)$, $(-3, 2)$, with the right angle at the former point. Find the locus of the third vertex.

21. Two sides of a parallelogram are the lines $y = 3x + 1$, $y = 2x + 2$; the other two sides intersect at $(-2, 3)$. Find the area of the parallelogram.

22. The base of an isosceles triangle joins the points $(-1, 3)$, $(4, 8)$; the vertex lies on Ox. Find the vertex.

23. One side of a rectangle joins the points $(0, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{3}{4})$. Find the loci of the other vertices.

24. A circle touches the line $y = 8x + 3$ at $(-1, -5)$. Find the locus of its center.

25. Can a circle be drawn touching the lines $y = 2x + 4$ and $y = -x + 2$ at $(-4, -4)$ and $(5, -3)$ respectively?

26. Prove that the medians of a triangle are concurrent.

34. The general equation of the first degree. If an equation is of the first degree in x and y, it can contain at most a term in x, a term in y, and a constant term: i.e. it can be written in the form

$$Ax + By + C = 0. \tag{3}$$

Now if $B = 0$, the equation may be written

$$x = -\frac{C}{A},$$

which represents a line parallel to Oy. If $B \neq 0$, the equation can be solved for y: i.e. it can be written in the form

$$y = mx + b,$$

which by § 33 represents a line. Hence we have proved the important

THEOREM: *The locus of every equation of the first degree is a straight line.*

The converse of this theorem is also true:

The equation of a straight line is always of the first degree.

For, if the line is parallel to Oy, its equation is of the form

$$x = k,$$

which is of the first degree. If it intersects the y-axis, it must have a certain slope and a certain y-intercept (either or both may of course be 0), and by § 33 its equation may be written

$$y = mx + b,$$

which is also of the first degree.

As suggested above, to reduce the equation of any line to the slope form (except a line parallel to Oy, to which the slope form does not apply), we have only to *solve the equation for* y.

A simple application of the slope form is furnished by the following

Example: Find the equation of a line through $(3, -1)$ perpendicular to the line $3x + 2y = 6$.

The given equation when reduced to the slope form becomes

$$y = -\tfrac{3}{2}x + 3.$$

Thus the slope of the given line is $-\tfrac{3}{2}$, whence that of the perpendicular must be $\tfrac{2}{3}$. The equation of a line of slope $\tfrac{2}{3}$ through $(3, -1)$ is

$$y + 1 = \tfrac{2}{3}(x - 3),$$

which reduces to

$$2x - 3y = 9.$$

35. Parallel and perpendicular lines. By reduction to the slope form, it is easily seen that the lines

$$Ax + By + C = 0,$$
$$Ax + By + K = 0$$

are parallel, while the lines

$$Ax + By + C = 0,$$
$$Bx - Ay + K = 0$$

are perpendicular. Hence, if a line is to be parallel to a given line, the coefficients of x and y in the required equation may be taken *the same as those in the given equation;* if a line is to be perpendicular to a given line, the coefficients of x and y in the required equation may be found by *interchanging the coefficients of x and y and changing the sign of one of them.* In each case, of course, the constant term must be determined by an additional condition.

Example: Write the equation of a line through $(3, -1)$ perpendicular to the line $3x + 2y = 6$.

The left member of the required equation will be $2x - 3y$; if the new equation is to be satisfied by the coördinates $(3, -1)$, the right member must be what the left member becomes when those coördinates are substituted. Hence the answer is

$$2x - 3y = 6 + 3,$$

or

$$2x - 3y = 9.$$

EXERCISES

Reduce the following equations to the slope form, and draw the lines.

1. $3x - y + 6 = 0$.

2. $2x + 5y = 0$.

3. $x = 5y + 3$.

4. $3x + 7y + 6 = 0$.

5. $3y + 4 = 0$.

6. $3 - 5y = 0$.

7. $\sqrt{3}x - y = 2$.

8. $\frac{2}{3}x + \frac{3}{5}y + \frac{1}{2} = 0$.

Find the equations of the following lines.

9. Through $(0, 0)$ perpendicular to the line $3x - 5y + 6 = 0$.

10. Through $(5, -2)$ perpendicular to the line $x + 6y = 3$.

11. Through $(-3, -4)$ perpendicular to the line $6x - y = 1$.

12. Through $(-\frac{1}{12}, \frac{1}{10})$ perpendicular to the line $\frac{3}{4}x - \frac{5}{3}y + \frac{11}{8} = 0$.

13. Through $(2, 0)$ perpendicular to the line $4x = 7$.

14. Through $(-1, 5)$ perpendicular to the line $5y + 8 = 0$.

15. Through $(\frac{3}{2}, -2)$ parallel to the line $3x - 2y + 5 = 0$.

16. Through $(1, 6)$ parallel to the line $y = 5x - 5$.

17. Through $(2, 0)$ parallel to the line $4x - 3y = 7$.

18. Through $(-3, -1)$ parallel to the line $4x = 5y + 6$.

19. Through $(2, -3)$ parallel to the line $3x - 5 = 0$.

20. Erect a perpendicular to the line $3x + 2y = 5$ at its point of intersection with the line $2x + 6y + 9 = 0$.

21. Write the equations of the lines (*a*) parallel, (*b*) perpendicular to the line $3y + 2 = 0$ and passing at a distance 2 from the point $(5, -3)$.

22. In the equation $Ax + By + C = 0$, what is the geometric meaning of the condition $A = 0$? $B = 0$? $C = 0$?

23. Two vertices of a right triangle are $(2, 1)$, $(3, -3)$, with the right angle at the former point. Find the locus of the third vertex.

24. One side of a rectangle is the line from $(5, -6)$ to $(-2, -3)$; find the loci of the other vertices.

25. Find the area of the triangle whose sides are $x = 1$, $3x - 2y = 8$, $2x + 3y = 1$. Solve by two methods.

26. Trace the curve $x^2 - 2xy + y^2 = x - y$.

27. Trace the curve $8x^2 - 10xy - 3y^2 = 0$.

28. Prove that the lines $x + 2y + 1 = 0$, $6x - 3y - 5 = 0$, $y = 2x - 1$, and $4x + 8y + 7 = 0$ form a rectangle.

36. Equations containing arbitrary constants; families of curves. An equation containing an *arbitrary constant* — i.e. a constant to which any value may be assigned at pleasure — may be thought of as representing an infinite number of curves, since infinitely many values may be assigned to the constant and each value determines a definite curve. Such a system is often spoken of as a *family* of curves.

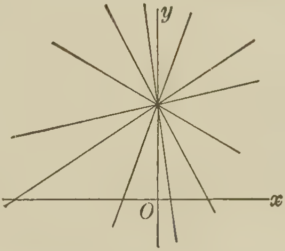


FIG. 30

For instance, the equation

$$y = mx + 3,$$

where m is arbitrary, represents the family of straight lines (Fig. 30) that intersect the y-axis at (0, 3); the equation

$$x^2 + y^2 = a^2,$$

where a is arbitrary, represents all the circles that can be drawn with center at O.

From a slightly different point of view, an equation containing an arbitrary constant may be considered as representing a single curve that is not as yet completely determined. Such a curve may be made to satisfy an additional condition of some sort: for example, to pass through a given point. The coördinates of this point may be substituted for x and y in the equation of the curve; this gives an equation that determines the constant, whose value may then be substituted back in the original equation. Evidently any condition whatever may be imposed upon the curve, provided that this condition when expressed analytically gives an equation which determines the constant.

For instance, to determine that one of the family of lines

$$y = mx + 3$$

that passes through (1, 7), we substitute the coördinates 1, 7 for x and y:

$$7 = m + 3,$$

whence

$$m = 4,$$

and the answer is

$$y = 4x + 3.$$

More generally, an equation may involve two or more arbitrary constants: e.g.

$$y = mx + b,$$

$$y = ax^2 + bx + c.$$

The number of arbitrarily assigned points through which a curve may be made to pass, or in other words, *the number of points required to determine the curve, is equal to the number* *of independent constants in the equation of the curve.* For upon substituting the coördinates of the points in turn, the number of equations obtained involving the constants is in general just equal to the number of constants; i.e. we have just enough equations to determine the unknowns. Thus the number of points required to determine a curve may be found by merely counting the number of independent constants in the equation of the curve.

However, in counting the number of constants, care must be taken to make sure that all of them are really "essential" — i.e. that the number appearing cannot be replaced by a smaller number without loss of generality. An illustration of what is meant appears in the second paragraph of the next article.

37. Straight line determined by two conditions. The equation of a straight line in the slope form, viz.

$$y = mx + b,$$

contains two constants, m and b, so that *a straight line is determined by two points or other conditions.*

The general equation of the straight line

$$Ax + By + C = 0$$

appears at first sight to contain three constants, A, B, C, which would lead us to suppose that a line is determined by three points. But we may divide the equation through by any one of the constants which is not 0, thus introducing the *ratios* of the other two constants to this one, and these two ratios are the only essential constants appearing.

EXERCISES

Write the equations of the following families of lines.

1. Parallel to $3x + 5y + 7 = 0$.

2. Having the y-intercept 2.

3. Having the slope $\frac{2}{3}$.

4. Perpendicular to the line $x - 3y = 6$; determine that one of the family that passes through $(5, -1)$.

5. Having the x-intercept 6.

6. Passing through $(-1, -3)$; determine that one of the family that passes through $(-2, 5)$.

7. Passing through $(5, -2)$; determine that one of the family that crosses the line $3x + 8y + 7 = 0$ at right angles.

8. Parallel to the y-axis.

9. By direct substitution, determine m and b so that the line $y = mx + b$ shall pass through $(2, 4)$ and $(-3, -5)$.

10. For what value of A does the line $Ax + 3y + 1 = 0$ pass through $(2, 2)$? Have a slope 3? A y-intercept 2?

11. Determine B so that the line $x + By + 3 = 0$ shall have a y-intercept 6.

12. Express analytically the condition that the line $y = mx + b$ shall pass through (x_1, y_1).

How many points are required to determine the following curves?

13. $3x + 2y + C = 0.$ **14.** $Ax + By = 0.$

15. $x^2 + y^2 = 2ax.$ **16.** $y = ax^2 + bx + c.$

17. $Ax^2 + Bxy + Cx^2 + Dx + Ey + F = 0.$

18. $y = ax^3 + bx^2 + cx + d.$

19. $ax^2 + bx + c = 0.$ What is represented by this equation?

20. Determine a and b so that the curve $y = ax^2 + bx$ shall pass through $(2, 1)$, $(1, 4)$. Plot the curve.

21. Make the curve $y = ax^3 + bx^2 + cx + d$ pass through the points $(0, 0)$, $(-1, 5)$, $(1, -11)$, $(2, -22)$, and plot the curve.

38. The intercept form. Let the intercepts of a straight line on the axes be $OA = a$ and $OB = b$ respectively. Choosing a point $P\colon (x, y)$ in a general position on the line, we see that the triangles MAP and OAB are similar. Whence

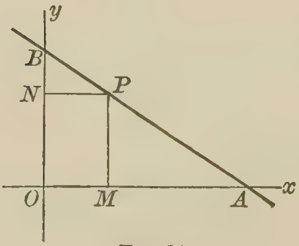


FIG. 31

$$\frac{MP}{OB} = \frac{MA}{OA},$$

or

$$\frac{y}{b} = \frac{a - x}{a}.$$

This is the equation of the line; it is usually written in the equivalent form

$$\frac{x}{a}+\frac{y}{b}=1. \tag{4}$$

This very important formula is called the *intercept form* of the equation of the straight line; it is especially useful in problems in which the intercepts of a line are given, or are involved in the data in some way.

To write the equation of any line in the intercept form we have merely to find the intercepts and substitute in (4). This form evidently fails in case the line passes through the origin or is parallel to either axis.

EXERCISES

Reduce the following equations to the intercept form, and draw the lines.

1. $3x-2y+7=0$. **2.** $2x-7y=8$.

3. $\frac{2}{3}x+\frac{3}{4}y=1$. **4.** $12x+15y+2=0$.

5. Derive the intercept form from the fact that, in Fig. 31, the area of the rectangle plus the area of the small triangles equals the area of the large triangle.

6. Apply the method of Ex. 5 to a figure in which a is negative, b positive, and P lies in the third quadrant.

7. Derive the intercept form by the method of § 38, using a figure in which a and b are both negative and P lies in the second quadrant.

8. Derive the intercept form from the fact that the area of the triangle APB is 0.

9. Derive the intercept form from the fact that the segments AP, PB have the same slope.

10. Derive the intercept form from the fact that $AP+PB=AB$.

11. A line makes equal intercepts on the axes and passes through (3, 2). Find its equation.

12. Solve Ex. 11 in another way.

13. A line makes equal intercepts on the axes and passes through (2, − 1). Find its equation.

14. The intercepts of a line are numerically equal but of opposite sign; the line passes through $(1, -5)$. Find its equation by two methods.

15. A line makes equal intercepts on the axes and passes at a distance 3 from the origin. Find its equation.

16. Write the equation of the family of straight lines for which the y-intercept is twice the x-intercept. Determine that member of the family that passes through $(7, 3)$.

17. The x-intercept of a line is -5 and the line passes at a distance 4 from O. Find its equation.

18. Determine a so that the line $\frac{x}{a} + \frac{y}{-5} = 1$ shall intersect the line $2x + 3y = 8$ at right angles.

19. What is the condition that the line $\frac{x}{a} + \frac{y}{b} = 1$ shall pass through $(1, 2)$? Through (x_1, y_1)?

20. A circle touches the lines $x + 3y = 6$, $2x + 6y + 3 = 0$. Find the locus of its center.

21. A circle touches the lines $\frac{3}{10}x - \frac{1}{2}y + \frac{6}{5} = 0$, $\frac{1}{2}x - \frac{5}{6}y + \frac{1}{3} = 0$. Find the locus of its center.

22. A rectangle is inscribed in a right triangle of base b and height h. What relation must hold between the base and altitude of the rectangle?

23. A right circular cylinder is inscribed in a right circular cone of radius r and height h. What relation must hold between the radius and height of the cylinder?

24. In Ex. 23, find the volume of the cylinder if its height is half that of the cone.

39. The normal form. A straight line is determined if we know the length p of the perpendicular from the origin to the line, together with the angle β which this perpendicular makes with Ox. To find the equation of a line

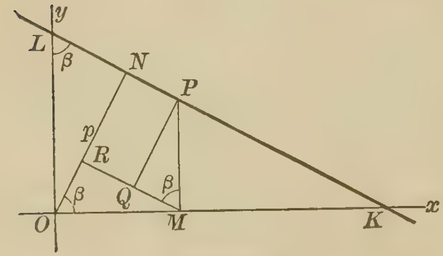


FIG. 32

determined in this way, assume a point $P: (x, y)$ on the line (the line KPL in Fig. 32) and draw auxiliary lines as shown. Now

$$OR + RN = ON,$$

or

$$OR + QP = ON.$$

But

$$OR = x \cos \beta, \quad QP = y \sin \beta, \quad ON = p,$$

so that

$$(5) \qquad \boldsymbol{x \cos \beta + y \sin \beta = p.}$$

This is the *normal form* of the equation of the straight line. We shall find it very useful in problems involving the distance of a point from a line, the distance between parallel lines, etc.

For definiteness we shall adopt the conventions that p is always positive and β is a positive angle less than 360°.

To reduce the general equation

$$(1) \qquad Ax + By + C = 0$$

to the normal form, we note that the slope of this line is $-\frac{A}{B}$, so that the slope of a line perpendicular to this is $\frac{B}{A}$. But the perpendicular makes the angle β with Ox: thus

$$\tan \beta = \frac{B}{A},$$

whence

$$\cos \beta = \frac{A}{\pm \sqrt{A^2 + B^2}}, \qquad \sin \beta = \frac{B}{\pm \sqrt{A^2 + B^2}}.$$

This shows that to reduce (1) to the normal form we must divide through by $\pm \sqrt{A^2 + B^2}$, choosing signs so that p shall be positive. In words, the rule is as follows:

To reduce the equation of any line to the normal form, divide through by the square root of the sum of the squares of the coefficients of x and y, and choose signs so that the constant term is positive in the right member.

40. Distance between parallel lines; distance of a point from a line. To find the distance between two parallel lines we evidently need only to reduce the equations of the lines to the normal form, and take the sum or the difference of the p's according as the lines are on opposite sides, or the same side, of the origin.

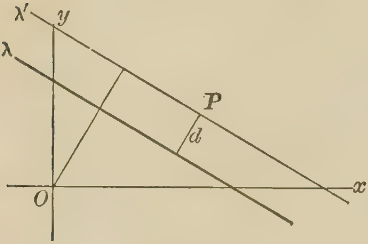


FIG. 33

To find the distance of a point P from a line λ, we write the equation of a line λ' through P parallel to λ, and then find the distance d between λ and λ'.

EXERCISES

1. Reduce the following equations to the normal form, and find the distance of each line from the origin:

(*a*) $3x - 4y - 8 = 0$; (*b*) $4x + y + 2 = 0$;
(*c*) $2x - 3y + 6 = 0$; (*d*) $x + 2y = 0$;
(*e*) $y = 5$; (*f*) $2x + 7 = 0$.

2. Write the equation of a line if its distance from the origin is 5 and the perpendicular upon it from the origin makes with Ox an angle of (*a*) 0°; (*b*) 60°; (*c*) 90°.

3. Solve Ex. 2 if the angle is (*a*) 120°; (*b*) 180°; (*c*) 270°; (*d*) 315°.

4. A line makes an angle of 45° with Ox and passes at a distance 3 from the origin. Find its equation. *Ans.* $y = x \pm 3\sqrt{2}$.

5. A line passes at a distance 4 from the origin and its slope is -5. Find its equation.

6. In Fig. 32, what are the coördinates of K? Of L? Solve by trigonometry directly from the figure, without using the equation of the line. *Ans.* K: $(p \sec \beta, 0)$; L: $(0, p \csc \beta)$.

7. Using the result of Ex. 6, derive the normal form from the fact that the area of the triangle KPL is 0.

8. Derive the normal form by use of the intercept form.

9. Derive the normal form by means of similar triangles.

10. Derive the normal form by equality of areas.

11. In Fig. 32, what are the coördinates of N?

12. Derive the normal form from the fact that $\overline{ON}^2 + \overline{NP}^2 = \overline{OP}^2$.

13. Derive the normal form from the fact that ON is perpendicular to NP.

14. Derive the normal form from the fact that the midpoint of OP is equidistant from O and N.

Find the equations of the following lines.

15. Parallel to $x + 3y + 2 = 0$ and passing at a distance 3 from the origin. *Ans.* $x + 3y = \pm 3\sqrt{10}$.

16. Parallel to $3x + 6y = 7$ and passing at a distance 5 from O.

17. Parallel to $2x - y = 6$ and (*a*) half as far from O; (*b*) three times as far from O; (*c*) 2 units farther from O.

18. Parallel to $x - y = 1$ and at a distance $2\sqrt{2}$ from that line.

19. Parallel to $3x - 2y = 6$ and passing at a distance 2 from the point (5, 3). *Ans.* $3x - 2y = 9 \pm 2\sqrt{13}$.

20. Parallel to $5x + 12y + 1 = 0$ and passing at a distance 2 from (0, 6).

21. Find the locus of points at the distance 4 from the line $4x + 3y + 6 = 0$.

22. A circle of radius 5 touches the line $2x - y + 3 = 0$. Find the locus of its center.

23. One side of a square is the line from (0, 5) to (3, 1). Find the other vertices.

24. Two sides of a square of area 9 lie in the lines $3x - 4y = 0$, $4x + 3y + 10 = 0$. Find the vertices.

25. The base of a triangle is the line from (6, 4) to (5, 2); the area is $\sqrt{10}$. Find the locus of the vertex by two methods.

Ans. $2x - y = 8 \pm 2\sqrt{10}$.

26. In Ex. 25, if the triangle is isosceles, find the vertex.

Ans. $(\frac{11}{2} \pm \frac{4}{5}\sqrt{10}, 3 \mp \frac{2}{5}\sqrt{10})$.

27. The base of an isosceles triangle is the line from (0, 0) to (8, − 6); the equal sides are of length 13. Find the vertex.

28. A point is equidistant from the points (4, 4) and (5, 7), and is at a distance $3\sqrt{2}$ from the line $2x - 2y + 3 = 0$. Find its coördinates.

Find the distances between the following pairs of lines.

29. $2x - 5y = 3,\ 2x - 5y = 6.$ *Ans.* $\frac{3}{29}\sqrt{29}$.

30. $x + 3y + 7 = 0,\ x + 3y - 4 = 0.$

31. $2x - y = 10,\ 6x - 3y + 1 = 0.$

32. $x + 5y + 4 = 0,\ 2x + 10y + 7 = 0.$

33. Two sides of a square lie in the lines $x + y = 0$, $x + y + 6 = 0$. Find the area of the square.

34. Find the area of the rectangle whose sides lie in the lines $x - 6y + 3 = 0$, $x - 6y + 1 = 0$, $6x + y = 5$, $12x + 2y + 3 = 0$.

Find the distances between the following points and lines.

35. $(5, 1),\ 3x + 2y + 6 = 0.$ *Ans.* $\frac{23}{13}\sqrt{13}$.

36. $(0, -3),\ x + 2y = 3.$

37. $(4, 1),\ x + 2y = 6.$

38. $(0, 0),\ 2x - y + \frac{3}{2} = 0.$

39. $(4, 2),\ 3y + 4 = 0.$

40. Determine p so that the line $\frac{3}{5}x - \frac{4}{5}y = p$ shall pass through $(2, -5)$.

41. By two methods, find the area of the triangle whose vertices are (0, 0), (5, 6), (3, 2).

42. A circle touches the lines $3x - 5y + 7 = 0$, $3x - 5y = 9$. Find its radius and the locus of its center.

43. Write the equation of the family of lines tangent to the circle of radius a with center at O.

44. A circle of radius 5 is drawn with center at O. Find the tangents to this circle through (11, 2) by writing the equation of any line through (11, 2) and making the line pass at a distance 5 from O.

Ans. $3x - 4y = 25$, $7x + 24y = 125$.

41. A formula for the distance of a point from a line. Given a line λ (Fig. 34) whose equation in the normal form is

$$x \cos\beta + y \sin\beta = p,$$

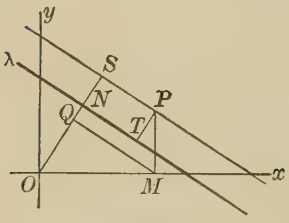


FIG. 34

and a point P: (x, y) not on that line, it is clear that the expression $x \cos\beta + y \sin\beta$ represents the projection of OMP upon the perpendicular to λ through O: i.e.

$$x \cos\beta + y \sin\beta = OQ + QS = OS.$$

Hence

$$x \cos \beta + y \sin \beta - p = OS - ON = NS = TP;$$

or, *the distance of the point* (x, y) *from the line*

$$x \cos \beta + y \sin \beta - p = 0$$

is given by the formula

$$\boldsymbol{d = x \cos \beta + y \sin \beta - p.} \tag{6}$$

This formula might of course be used instead of the method of § 40. It is clear that the distance as given by the formula will be positive if the point and the origin are on opposite sides of the line, negative if they are on the same side.

42. Angle bisectors. The bisector of the angle between two lines is the *locus of points equidistant from the two lines.* Given the equations of two lines λ_1, λ_2 in the normal form,

$$x \cos \beta_1 + y \sin \beta_1 = p_1,$$
$$x \cos \beta_2 + y \sin \beta_2 = p_2,$$

let us assume a point P: (x, y) in the bisector of that angle in which the origin lies. Then the distances M_1P and M_2P are numerically equal; they are also algebraically equal since in both cases O and P are on the same side of the line and both distances are negative, or when P is in the position P' both are positive. But, by the result of the preceding paragraph,

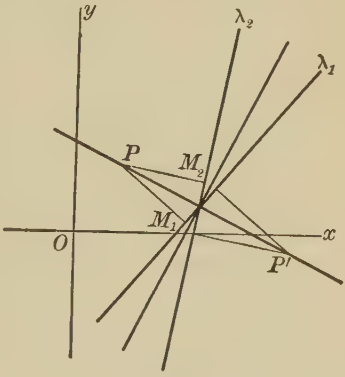


FIG. 35

$$M_1P = x \cos \beta_1 + y \sin \beta_1 - p_1,$$
$$M_2P = x \cos \beta_2 + y \sin \beta_2 - p_2,$$

so that the equation of the locus of P is

$$x \cos \beta_1 + y \sin \beta_1 - p_1 = x \cos \beta_2 + y \sin \beta_2 - p_2.$$

In a similar way the equation of the other angle bisector is found to be

$$x \cos \beta_1 + y \sin \beta_1 - p_1 = -(x \cos \beta_2 + y \sin \beta_2 - p_2).$$

In summary, we have the following result:

The equations of the lines bisecting the angles between the lines

$$x \cos \beta_1 + y \sin \beta_1 = p_1,$$
$$x \cos \beta_2 + y \sin \beta_2 = p_2$$

are

$$x \cos \beta_1 + y \sin \beta_1 - p_1 = \pm (x \cos \beta_2 + y \sin \beta_2 - p_2).$$

If neither of the given lines passes through the origin, the bisector of the angle in which the origin lies is found by using the positive sign.

EXERCISES

1. A point moves so as to remain equidistant from the point (1, 0) and the line $x - y = 0$. Find the equation of its locus.
Ans. $x^2 + 2xy + y^2 - 4x + 2 = 0$.

2. A point moves so as to remain equidistant from the origin and the line $x + 4y = 3$. Find the equation of its locus.
Ans. $16x^2 - 8xy + y^2 + 6x + 24y - 9 = 0$.

3. A circle passes through the point (1, 1) and touches the line $3x - y = 6$. Find the locus of its center.

4. A point moves so that its distance from the line $x + y = 3$ is 4 times its distance from the line $7x - y + 1 = 0$. Find its locus.

Find the bisectors of the angles between the following lines.

5. $3x + y + 6 = 0$, $2x + 6y = 5$.
Ans. $8x + 8y + 7 = 0$, $4x - 4y + 17 = 0$.

6. $4x - 5y = 6$, $x + 7y = 8$.

7. $3x - y = 0$, $x + y + 1 = 0$.

8. $x = 5$, $3x - 4y = 0$.

9. $x = 0$, $y = 6$.

10. A circle touches the lines $2x + y = 5$, $x + 2y = 3$. Find the locus of its center.

11. A circle touches the lines $3x + 2y = 3$, $7x - 2y + 6 = 0$. Find the locus of its center.

12. Find the center of the circle inscribed in the triangle formed by the lines $x + y = 3$, $7x - y = 10$, $x - y + 6 = 0$.

13. Prove analytically that the bisectors of the angles formed by two lines are perpendicular.

14. What result is obtained by applying the theorem of § 42 to two parallel lines?

15. Solve Ex. 20, p. 56, by the method of § 42.

16. A point moves so that the sum of its distances from the lines $x - y + 3 = 0$, $x + y = 0$ is equal to $\sqrt{2}$. Find the equation of its locus, and trace the curve.

17. If a point moves so that the sum of its distances from two fixed lines is constant, prove that its locus is a straight line.

18. A point moves so that its distances from two fixed lines bear a constant ratio to each other. Prove that its locus is a straight line.

19. A point moves so that the product of its distances from two parallel lines is constant. Prove that its locus consists of two lines parallel to the given lines.

43. Angle between two lines. By the *angle from* a line λ_1 *to* another line λ_2 we shall understand the *acute angle* through which λ_1 must be rotated to come to coincidence with λ_2. This angle will be considered positive if measured counter-clockwise, negative clockwise. In case we are concerned only with the magnitude of the angle without regard to sign, we may speak merely of the angle *between* the lines.

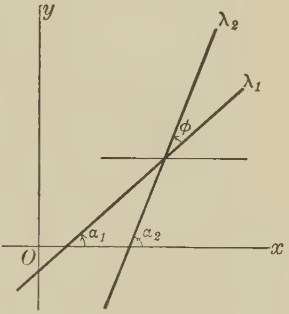


FIG. 36

Let the lines λ_1, λ_2 make the (positive or negative) acute angles α_1, α_2 with Ox, and denote by ϕ the angle from λ_1 to λ_2. Then we have

$$\phi = \alpha_2 - \alpha_1,$$

and

$$\tan\phi = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}.$$

But the slopes of the lines are

$$\tan \alpha_1 = m_1,\ \tan \alpha_2 = m_2,$$

so that *the angle between* two lines of slopes* m_1, m_2 *is given by the formula*

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{7}$$

This formula evidently fails if either of the lines is parallel to Oy, but in that case the required angle is readily found directly.

An important application of (7) is to find the slope of a line making a given angle with a given line, as in the following

Example: Find the equations of the lines passing through P: (5, 3) and making with the line

$$3x + y = 6$$

an angle† $\phi = \arctan 2$.

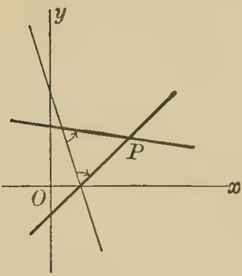


FIG. 37

For definiteness, let us measure ϕ from the given to the required line. Then $\tan \phi = 2$, $m_1 = -3$, so that, when ϕ is measured in the positive sense,

$$2 = \frac{m_2 + 3}{1 - 3m_2},$$

whence $m_2 = -\frac{1}{7}$, and the desired equation is

$$y - 3 = -\tfrac{1}{7}(x - 5),$$

or

$$x + 7y = 26.$$

Similarly, measuring ϕ in the negative sense, we write

$$-2 = \frac{m_2 + 3}{1 - 3m_2},$$

---

* More precisely, the angle from the first line to the second.

† The symbol $\arctan k$ (also written $\tan^{-1} k$) means the *angle whose tangent is k:* i.e. the statement

$$\phi = \arctan k$$

is equivalent to the statement

$$\tan \phi = k.$$

so that $m_2 = 1$, and the second required line is

$$y - 3 = x - 5,$$

or

$$x - y = 2.$$

44. Infinite roots of an algebraic equation. Given an algebraic equation of the nth degree in one unknown,

$$(1) \qquad a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

let us put $x = \frac{1}{z}$:

$$\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n = 0,$$

or after clearing of fractions,

$$(2) \qquad a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = 0.$$

Now in equation (2), if a_0 approaches 0, one value of z evidently approaches 0, and the corresponding value of x increases indefinitely, since $x = \frac{1}{z}$; or in other words, when a_0 approaches 0, one root of equation (1) becomes infinite. Further, if both a_0 and a_1 approach 0, two roots of (2) become 0, and two roots of (1) become infinite; etc. These results are expressed by the following

THEOREM: *In an algebraic equation in one unknown, if the first coefficient* * *becomes* 0, *one root becomes infinite; if the first two coefficients become* 0, *two roots become infinite; etc.*

A typical application of this theorem is furnished by the following

Example: Find the equation of a straight line through the point P: (1, 1) making with the line $x - 2y = 3$ an angle $\phi = \arctan 2$.

---

* That is, the coefficient of the highest power.

The slope of the given line is $\frac{1}{2}$; hence, when ϕ is measured as in the figure,

$$(3) \qquad 2 = \frac{m_2 - \frac{1}{2}}{1 + \frac{1}{2} m_2},$$

$$2 + m_2 = m_2 - \tfrac{1}{2},$$

or

$$(4) \qquad \tfrac{5}{2} = 0.$$

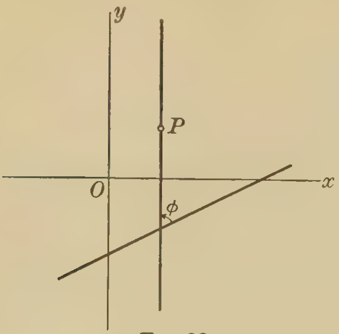


FIG. 38

This result, absurd when taken literally, can be interpreted in view of the fact, *known beforehand*, that equation (3) must be a linear equation in m_2: we conclude from (4) that the root is infinite, or in other words that the required line is parallel to Oy. Its equation is therefore

$$x = 1.$$

EXERCISES

Find the angle between the given lines.

1. $y = 2x - 1, \quad y = 5x + 4.$

2. $3x - 2y = 0, \quad 4x + 3y + 5 = 0.$

3. $x + 4y = 5, \quad 12y = 1 - 3x.$

4. $y + 5x = 5, \quad 2x - 5y = 4.$

5. $3x + y = 6, \quad 2x - 3y + 7 = 0.$

6. $2x - 4y = 5, \quad 2x + y + 3 = 0.$

7. $x = 6y, \quad x + 2y = 7.$

8. $2x + 3 = 0, \quad 3x - 5y - 5 = 0.$

9. Find the area of the triangle whose vertices are (3, 4), (2, 3), (1, − 2), using the formula

$$A = \tfrac{1}{2} bc \sin \alpha,$$

where α is the angle between the sides b, c.

10. Using the formula of Ex. 9, find the area of the triangle whose vertices are (− 1, 5), (4, 2), (1, − 3).

11. Find the equations of the lines through the point (6, − 2) making with the line $x + 6y + 5 = 0$ an angle $\phi = \arctan 3$.

12. Find the equations of the lines through the origin making with the line $7x + 2y = 1$ an angle arctan $\frac{1}{3}$.

13. The base of an isosceles triangle is the line from (6, 3) to (7, 1); the altitude is $\frac{1}{3}\sqrt{5}$. Find the third vertex by using formula (7). Describe several other methods for solving this problem.

14. One side of a square is the line from (0, 5) to (3, 1). Find the other vertices by a new method. (Ex. 23, p. 59.)

15. Solve Ex. 27, p. 59, in a new way.

16. Find the equations of the straight lines through $(9, -6)$ passing at a distance 1 from the point $(4, -1)$.

17. Find the lines through (3, 2) passing at a distance $\sqrt{5}$ from (5, 3).

18. One side of an isosceles right triangle is the line from $(-2, 16)$ to $(-10, 10)$, with the right angle at $(-2, 16)$. Find the third vertex by several methods.

19. Solve Ex. 44, p. 60, by a new method.

20. Find the equations of the lines through (0, 0) making with the line $4x - y = 1$ the angle arctan $\frac{1}{4}$.

21. Find the equations of the lines through $(2, -3)$ making with the line $3x - 5y = 6$ the angle arctan $\frac{5}{3}$.

22. A circle of radius $\sqrt{2}$ is drawn with center at (6, 4). By two methods, find the tangents to this circle through the origin.

23. A circle of radius 2 is drawn with center at (3, 1). By two methods, find the tangents to this circle through $(1, -5)$.

24. What becomes of the straight line $Ax + By + C = 0$ when A approaches 0? When both A and B approach 0?

25. Solve the quadratic equation $ax^2 + bx + c = 0$ and prove directly that one root becomes infinite as a approaches 0, two roots become infinite when both a and b approach 0.

CHAPTER V

THE CIRCLE

45. Definitions; standard forms. A *circle* is the locus of a point that moves at a constant distance from a fixed point. The fixed point is called the *center*, and the constant distance is the *radius*. The radius as thus defined is of course merely a number of linear units; the term is also used, as in elementary geometry, to mean a line-segment joining the center and a point of the curve. A *diameter* of a circle may mean either a straight line through the center, or the segment of such a line lying inside the curve.

From the definition the equation of a circle with given center and radius may be written down at once. For the circle with center at O and radius a, we have for any position of the moving point

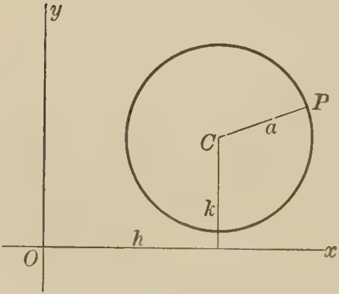


FIG. 39

$$\sqrt{x^2 + y^2} = a;$$

if the center is at any point C: (h, k), as in Fig. 39,

$$\sqrt{(x - h)^2 + (y - k)^2} = a.$$

Hence *the equation of a circle of radius a is, if the center is the point* $(0, 0)$,

$$x^2 + y^2 = a^2; \tag{1}$$

if the center is the point (h, k),

$$(x - h)^2 + (y - k)^2 = a^2. \tag{2}$$

Equations (**1**) and (**2**) are called *standard forms* of the equation of the circle. Of course (**1**) is merely a special case of (**2**): the case $h = k = 0$.

46. General equation. It appears from (2) that the equation of a circle is always of the *second degree.*

The most general possible equation of the second degree in x and y may contain, at most, terms in x^2, xy, y^2, x, y, and a constant: i.e. it may be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Consider now the special case in which $A = C$ and $B = 0$:

$$\mathbf{(3)} \qquad \boldsymbol{Ax^2 + Ay^2 + Dx + Ey + F = 0.}$$

We may evidently always divide this equation through by* A, transpose the constant term to the right member, and complete the squares in x and y; the equation then has the form

$$(x - h)^2 + (y - k)^2 = a^2,$$

and consequently *represents a circle.* (If this statement is to hold in all cases, we must agree to consider the term "circle" as including certain limiting forms discussed in the next article.) Equation (3) is called the *general form* of the equation of the circle.

Conversely, the equation of every circle may evidently be put in the form (3). We have therefore the

THEOREM: *Every equation of the second degree in which x^2 and y^2 have equal coefficients and the xy-term is missing represents a circle; and conversely.*

Example: Find the center and radius of the circle

$$4x^2 + 4y^2 - 4x + 2y + 1 = 0.$$

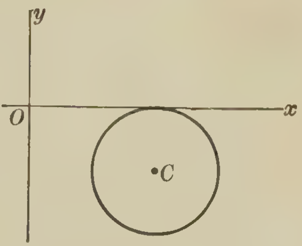


FIG. 40

First transpose the constant term to the right member and divide by 4:

$$x^2 + y^2 - x + \tfrac{1}{2}y = -\tfrac{1}{4}.$$

---

* If A were equal to 0, we should no longer have a second degree equation.

Then complete the squares in x and y:

$$x^2 - x + \tfrac{1}{4} + y^2 + \tfrac{1}{2}y + \tfrac{1}{16} = -\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{16},$$

or

$$(x - \tfrac{1}{2})^2 + (y + \tfrac{1}{4})^2 = \tfrac{1}{16}.$$

The center is the point $C: (\frac{1}{2}, -\frac{1}{4})$, and the radius is $\frac{1}{4}$.

47. Point circles; imaginary circles. It may happen that when the equation of a circle is reduced to the form (**2**) the right member becomes 0: i.e. the equation takes the form

$$(x - h)^2 + (y - k)^2 = 0.$$

This may be thought of as the equation of a circle with center at (h, k) and radius 0, a so-called "point circle": the locus is the single point (h, k). The fact that the locus is the point (h, k) also appears directly from the fact that the sum of two squares can equal 0 only when the two squares vanish separately, or in this case when $x = h$ and $y = k$.

Again, after reduction to the standard form the right member may be negative. Since the right member equals the square of the radius, the radius itself is imaginary in this case, and the equation represents an imaginary circle. The fact that there is no real locus appears also from the fact that in the domain of real numbers, the sum of two squares cannot be negative.

In most applications these cases are of little importance.

EXERCISES

Write the equations of the following circles; draw the figure in each case.

1. With center at $(0, 3)$ and radius 4.

2. With center at $(-1, -3)$ and radius 1.

3. With center at $(-\frac{1}{6}, \frac{1}{6})$ and radius $\frac{1}{6}$.

4. With center at $(2, 5)$ and touching the x-axis.

5. With center at $(-a, 0)$ and touching the y-axis.

6. With radius 5 and touching both axes.

7. With center at $(3, 3)$ and passing through the origin.

8. With center at (5, 2) and passing through (6, − 1).

9. With the points (2, 2) and (0, − 5) as ends of a diameter.

10. With center at (0, 0) and touching the line $3x - y = 6$.

11. With center at (4, 3) and touching the line $y = 2x$.

12. With radius 1 and center on the lines $\frac{2}{3}x + \frac{3}{4}y = 1$, $\frac{1}{2}x - \frac{1}{6}y = 2$.

✓ **13.** The hypotenuse of a right triangle is the line from (2, 3) to (5, − 1); the vertex lies on the x-axis. Find the vertex. (Two answers.)

14. Using equation (**2**), find the condition that a circle shall
(*a*) touch the x-axis;
(*b*) touch both axes;
(*c*) pass through the origin; $c = 0$
(*d*) have its center on the y-axis and pass through the origin;
(*e*) have its center on the line $y = x$;
(*f*) have its center on the line $3x + 2y = 5$;
(*g*) have its center on the line $x + y = 0$ and pass through the origin.

Draw the following circles.

15. $x^2 + y^2 + 4x - 8y = 0$.

16. $3x^2 + 3y^2 + 5x + 6y + 2 = 0$.

17. $x^2 + y^2 = 6x$.

18. $x^2 + y^2 + 5 = 6y$.

19. $6x^2 + 6y^2 - 2x - y = 10$.

20. $x^2 + y^2 - 2x + 4y + 5 = 0$.

21. $7x^2 + 7y^2 + 3x - y + 6 = 0$.

22. Find the locus of a point which moves so that the square of its distance from the origin is numerically equal to its distance from the line $x = k$. Draw the curve.

23. Find the locus of a point which moves so that the square of its distance from (3, 2) is numerically equal to its distance from the line $3x - 4y + 15 = 0$. Draw the curve.

24. A point moves so that the sum of the squares of its distances from the points (0, 0), (1, 0) is constant. Find its locus.

25. A point moves so that the sum of the squares of its distances from two fixed points is constant. Prove that its locus is a circle.

✓ **26.** A point moves so that its distance from the point (1, 0) is always twice its distance from the point (3, 5). Find the equation of its locus. *Ans.* $3x^2 + 3y^2 - 22x - 40y + 135 = 0$.

27. Find the locus of a point which moves so that its distances from two fixed points are in a constant ratio k. What kind of curve is described? Investigate the cases $k = 0$, $k = 1$.

Ans. $(1 - k^2)(x^2 + y^2) - 2(x_1 - k^2x_2)x - 2(y_1 - k^2y_2)y + x_1^2 + y_1^2 - k^2(x_2^2 + y_2^2) = 0$.

28. Derive equation (1), § 45, from the fact that an angle inscribed in a semicircle is a right angle.

✓ **29.** Prove analytically that an angle inscribed in a semicircle is a right angle.

30. Prove analytically that a circle is symmetric with respect to a diameter.

31. Prove analytically that two circles are symmetric with respect to their line of centers; also that two equal circles are symmetric with respect to the perpendicular bisector of their line of centers. (Write the single equation representing the two circles; cf. § 21.)

32. Prove that the circles $x^2 + y^2 - 10x - 8y - 4 = 0$, $x^2 + y^2 - 2x - 4y = 0$ are tangent.

33. Prove that the line $5x - 12y = 152$ is tangent to the circle $x^2 + y^2 - 3x + 10y - 15 = 0$, determine the point of contact, and find the equation of the other tangent parallel to this line.

48. Circle satisfying three conditions. We know from elementary geometry that a circle is determined by three points. This is proved analytically by the fact that the equation

$$(x - h)^2 + (y - k)^2 = a^2$$

contains three independent constants, h, k, and a.

More generally, a circle may be made to satisfy any three conditions which when expressed analytically lead to three simultaneous equations for determining the constants. Thus three tangents may be given, or two points and the radius, etc. It may not, however, be uniquely determined; there may be two or more circles satisfying the given conditions. For instance, we know that four different circles can be drawn touching three given lines.

The general method of finding the equation of a circle satisfying three conditions is, as suggested above, to express the conditions analytically by means of three simultaneous equations which may be solved for the constants. The conditions may be expressed most simply in some cases by using equation (**2**), in other cases by using (**3**). Very often, however, special methods may be devised which are shorter than the general method and are more easily interpreted geometrically. Such methods can usually be found by recalling the construction of elementary geometry for the center and radius and carrying out that construction analytically (cf. the second method below).

Example: Find the equation of the circle through the points P_1: (1, 1), P_2: (2, − 1), and P_3: (2, 3).

One method is to assume the equation of the circle in the form* (**3**), taking† $A = 1$:

$$x^2 + y^2 + Dx + Ey + F = 0.$$

Substituting the coördinates of the given points in turn, we obtain the equations

$$\begin{aligned} 2 + D + E + F &= 0, \\ 5 + 2D - E + F &= 0, \\ 13 + 2D + 3E + F &= 0; \end{aligned}$$

these equations may be solved for D, E, and F.

A second method, which is to be preferred on account of its more direct geometric interpretation, is as follows. The center lies on the perpendicular bisector of P_1P_2, whose equation is readily found to be $4x+2y=3$

$$2x - 4y = 3;$$

it also lies on the perpendicular bisector of P_1P_3, whose equation is

$$2x + 4y = 11.$$

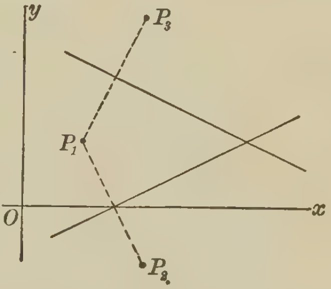


FIG. 41

---

* If we were to use (**2**) the equations obtained in h, k, and a would be of the second degree, and less convenient to solve.

† This involves no loss of generality, since the coefficient of x^2 and y^2 may always be made 1 by division.

The center is therefore the point of intersection of these lines, viz. $(\frac{7}{2}, 1)$. The radius is the distance from $(\frac{7}{2}, 1)$ to any one of the given points, which is found to be $\frac{5}{2}$. The equation of the circle is

$$(x - \tfrac{7}{2})^2 + (y - 1)^2 = \tfrac{25}{4},$$

or

$$x^2 + y^2 - 7x - 2y + 7 = 0.$$

Check:

$$\begin{aligned} 1 + 1 - 7 - 2 + 7 &= 0, \\ 4 + 1 - 14 + 2 + 7 &= 0, \\ 4 + 9 - 14 - 6 + 7 &= 0. \end{aligned}$$

EXERCISES

Find the equations of the following circles, drawing a figure in each case. Check the answers.

1. Through the points (3, 5), (– 1, – 4), (6, 3).

2. Through the points (2, 0), (0, 0), (1, 3).

3. Through the points (6, – 5), (4, – 3), (– 2, – 1).

4. Circumscribed about the triangle (0, 5), (4, – 3), (– 3, – 4).

5. Circumscribed about the triangle (3, 2), (1, – 1), (– 2, 1).

6. Through the points (2, 2), (4, 3), and having its center on the y-axis.

7. Through the points (1, 3), (5, – 2), and having its center on the line $6x + 5y = 10$.

8. Touching the line $x - 3y = 4$ at (1, – 1) and passing through (– 2, 8).

9. Touching the line $3x + 2y + 6 = 0$ at (– 2, 0) and passing through (4, – 1).

10. Touching the line $x - 5y + 3 = 0$ at (2, 1) and passing through $(-1, \frac{1}{2})$.

11. Touching the line $x + y = 6$ at (2, 4) and having a radius $\sqrt{2}$. Solve by two methods.

12. Touching the line $3x - y = 0$ at (– 1, – 3) and having a radius 5.

13. Touching the line $5x - 12y + 13 = 0$ at (7, 4) and having a radius 2.

14. Touching the lines $x + 3y = 5$, $x + 3y = 3$ and having its center on the line $2x + y + 1 = 0$.

15. Touching the lines $2x - y - 6 = 0$, $2x - y + 4 = 0$ and passing through the origin.

16. Touching the lines $x = 3$, $3x - 4y = 0$ and having its center on the line $x + 2y = 6$.

17. Touching the lines $x + 3y = 6$, $x - 3y = 10$ and having its center on the line $y = 2x$.

18. Inscribed in the triangle formed by the lines $3x - 4y = 5$, $4x - 3y + 10 = 0$, $y = 2$.

19. Inscribed in the triangle whose sides lie in the lines $3x + 4y = 5$, $y = 0$, $12x - 5y = 0$.

20. Touching the lines $x + 4y + 4 = 0$, $x + 4y = 2$, $y = 4x + 8$.

21. Having a radius $\sqrt{10}$, touching the line $2x + 6y = 3$, and having its center on Ox.

22. Having a radius $2\sqrt{2}$ and touching the lines $x + y = 0$, $x - y = 0$

23. Having a radius $\sqrt{5}$ and touching $x - 2y = 3$, $4x + 2y = 1$.
Ans. Centers at $(-\frac{11}{5}, -\frac{1}{10})$, $(\frac{9}{5}, \frac{19}{10})$, $(-\frac{1}{5}, -\frac{41}{10})$, $(\frac{19}{5}, -\frac{21}{10})$.

24. Having its center at (5, 4) and touching the circle $x^2 + y^2 = 4x$.

25. Touching the circle $x^2 + y^2 = 100$ at $(6, -8)$ and having a radius 12.

26. Tangent to the circles $x^2 + y^2 = 10x$, $x^2 + y^2 = -10x$ and having a radius 20.
Ans. $x^2 + (y \mp 10\sqrt{6})^2 = 400$; $x^2 + (y \mp 10\sqrt{2})^2 = 400$.

27. Having a radius $\sqrt{10}$, through the points (2, 2), (4, 4).

28. Having a radius $\frac{1}{2}\sqrt{5}$ and touching the lines $x + 2y = 3$, $2x + y = 4$.

29. Having a radius $\sqrt{26}$, through the points (1, 1), $(-7, -5)$.

30. Touching the circle $x^2 + y^2 + 2x - 6y + 5 = 0$ at (1, 2) and passing through $(5, -4)$.

49. Quadratic equations. A quadratic equation in one unknown x can contain, at most, terms in x^2, x, and a constant: every quadratic equation in x can therefore be put in the form

$$ax^2 + bx + c = 0.$$

Every quadratic equation has two roots. The student is assumed to be familiar with the usual methods of finding the roots: by inspection, by completing the square, and by use of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{4}$$

From formula (4) it appears that the two roots,

$$x_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$x_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a},$$

are

real and different if $b^2 - 4ac > 0$;

real and equal if $b^2 - 4ac = 0$;

imaginary if $b^2 - 4ac < 0$;

and conversely. The quantity $b^2 - 4ac$ is called the *discriminant* of the quadratic.

Example: Determine k so that the equation

$$2x^2 - 5kx + k^2 + 34 = 0$$

shall have equal roots.

The condition for equal roots is

$$b^2 - 4ac = 0,$$

or in this case

$$(-5k)^2 - 4 \cdot 2(k^2 + 34) = 0,$$

which reduces to

$$17k^2 = 272.$$

Whence

$$k = \pm 4.$$

If the first coefficient in a quadratic equation is unity, the equation can be written in the factored form

$$(x - x_1)(x - x_2) = 0,$$

where x_1 and x_2 are the roots. When expanded, this becomes

$$x^2 - (x_1 + x_2)x + x_1x_2 = 0.$$

Thus in a quadratic equation where the first coefficient is unity *the sum of the roots is equal to the second coefficient with the sign changed, and the product of the roots is the constant term.*

50. Intersection of a line and a circle; of two circles. To find the points of intersection of a line and a circle, we must evidently solve two simultaneous equations in x and y, of which one is linear and the other quadratic. The first step, of course, is to substitute in the quadratic equation the value of y (or x) from the linear equation; this gives a quadratic in x (or y) whose roots may be found; these roots when substituted back in the linear equation give the values of y (or x). According as the pairs of values of x and y thus obtained are real and different, real and the same, or imaginary, the line and the circle intersect in distinct real points, are tangent to each other, or do not intersect at all in real points.

To find the points of intersection of two circles we may eliminate x^2 and y^2 between the two equations, thus obtaining (unless the circles are concentric) a linear equation which may be solved with either of the original equations in the way described above.

EXERCISES

Solve the following equations by various methods. Check the results.

1. $x^2 - 7x - 8 = 0$.

2. $x^2 + x - 2 = 0$.

3. $3x^2 - 4x = 5$.

4. $2x^2 - 3x + 2 = 0$.

5. $2y - 3y^2 = 7$.

6. $4x^2 - x = 0$.

7. $m^2 + 8m + 16 = 0$.

8. $5x - 2x^2 = 0$.

Without solving, determine the nature of the roots of the following equations (whether real and different, real and equal, or imaginary); also find the sum and product of the roots.

9. $x^2 - 8x + 13 = 0$.

10. $x^2 + x + 1 = 0$.

11. $3x^2 + 4x - 1 = 0$.

12. $9x^2 - 6x + 1 = 0$.

13. Determine k so that the equation $x^2 - 3kx + 7 = 0$ shall have equal roots.

14. Determine k so that the equation $2x^2 - 5kx + k^2 - 3 = 4k$ shall have equal roots. *Ans.* $k = -\frac{16}{17} \pm \frac{2}{17}\sqrt{-38}$.

15. Determine m so that the equation $mx^2 - 3mx + 1 = 0$ shall have equal roots.

16. Determine k so that the sum of the roots of the equation $x^2 + kx - 2x + k^2 + 3 = 0$ shall be 0.

17. Without solving, show that for all values of a, one root of the equation $x^2 + ax + 1 = 0$ is the reciprocal of the other.

18. If the slopes of two lines are determined as the roots of the equation $m^2 + km - 1 = 0$, show that the lines are perpendicular for all values of k.

Find the points of intersection of the following lines and circles.

19. $x + 3y = 3$, $x^2 + y^2 = 37$.

20. $3x - 2y = 5$, $x^2 + y^2 - x + y - 20 = 0$.

21. $x^2 + y^2 = 13$, $x^2 + y^2 - 3x - 4y + 4 = 0$.

22. $2x^2 + 2y^2 + x = 0$, $3x^2 + 3y^2 - 5x - 4y = 0$.

23. $x^2 + y^2 = 9$, $x + y = 2$.

24. $x^2 + y^2 = 25$, $3x + 4y = 25$.

25. $x^2 + y^2 = a^2$, $x + y = a$.

26. Find the intercepts of the circle $x^2 + y^2 - 3x + 2y = 4$ on the coördinate axes.

27. Find the intercepts of the circle $x^2 + y^2 - 6x + 2y + 4 = 0$ on the axes.

28. Find the condition that the line $Ax + By + C = 0$ shall intersect the circle $x^2 + y^2 = a^2$ in two real points. Solve by two methods. *Ans.* $C^2 < a^2(A^2 + B^2)$.

29. Find the condition that the circles $(x - h)^2 + (y - k)^2 = b^2$, $x^2 + y^2 = a^2$ shall intersect in real points.

30. Find the equation of a circle of radius $2\sqrt{5}$ touching the circle $x^2 + y^2 = 45$ and having its center on the line $x + y = 3$.

Ans. $(x - \frac{3}{2} \mp \frac{1}{2}\sqrt{241})^2 + (y - \frac{3}{2} \pm \frac{1}{2}\sqrt{241})^2 = 20$,

$(x - 1)^2 + (y - 2)^2 = 20$, $(x - 2)^2 + (y - 1)^2 = 20$.

51. Tangents to plane curves. A straight line that intersects a curve in two or more distinct points is called a *chord*, or *secant.* The term "chord" is also used as in elementary geometry to denote a line-segment terminated by two points of a curve.

Let P be a fixed point of a plane curve, and P' a neighboring point. If P' be made to approach P along the curve, the secant PP' evidently approaches, in general, a definite limiting position (PT in the figure), and in this position, at the instant when P' coincides with P, it is called the *tangent to the curve at P*, or is said to *touch the curve at P.* The point P is the *point of contact.*

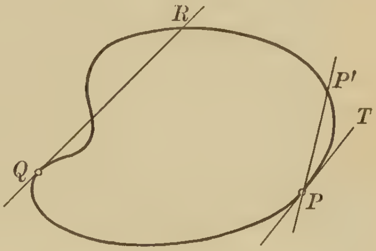


FIG. 42

The *slope of a curve* at any point is defined as the slope of the tangent at that point. Two curves intersecting in a point P are said to be *tangent at P* if they have the same slope — i.e. if they have a common tangent — at that point.

The *angle between two curves* at a point of intersection is the angle between their tangents at that point.

In elementary geometry a tangent to a circle is usually defined as a line which intersects the circle in one point. Such a definition would not hold for curves in general, as is shown by Fig. 42, where the line QR is both a secant and a tangent. The definition given above holds in general. Another reason why the definition given in elementary geometry is objectionable in our present work will be pointed out in the next article.

52. The condition for tangency. It is obvious from the definition of tangent that when the equation of a curve and that of a line tangent to it are solved as simultaneous equations, two of the pairs of values of x and y will be the same: i.e. *two of the points of intersection will coincide.*

Let us now confine our attention to curves of the *second degree*.* If the line and the curve are tangent, the quadratic in x (or y) obtained by substituting in the equation of the curve the value of y (or x) from the equation of the line will have equal roots: i.e. it will have

$$b^2 - 4\,ac = 0.$$

Conversely, *if this quadratic has equal roots the line is tangent to the curve*, since the points of intersection then have equal abscissas (or ordinates) and must coincide.† The condition

$$(5) \qquad \mathbf{b^2 - 4\,ac = 0}$$

applied to the quadratic obtained in this way is called the *condition for tangency* of the line and the curve.‡ If either the equation of the line or that of the curve contains an undetermined constant, the condition for tangency serves to determine that constant.

From what has been said it appears that the connection between algebra and geometry is much more clearly preserved if we think of a tangent as the limiting position of a secant in which the two points of intersection have come to coincidence, rather than as a line intersecting the curve in one point.

It is possible to solve problems involving tangents to circles by means of special methods based on the special properties of that curve, and we shall use such methods to some extent in the next few articles. On account of its generality, however, the method above described will also be emphasized.

---

* For the determination of tangents to higher curves the differential calculus is required.

† If the line is parallel to the y-axis the points have equal abscissas without necessarily coinciding. But in that case the equation of the line is $x = k$, so that in the process of solving the simultaneous equations we are led to a quadratic in y, and if this has equal roots the line is a tangent.

‡ This condition evidently applies only to curves of the second degree, since it is only in this case that the process outlined leads to a quadratic equation in x or y.

53. Tangents having a given slope. The way in which the theory of the preceding section may be applied to find tangents whose slope is given will be explained by means of the following

Example: Find the tangents to the circle $x^2 + y^2 = 1$ parallel to the line $y = x - \frac{1}{2}$.

The equation of any line parallel to $y = x - \frac{1}{2}$ is

(1) $y = x + k.$

We wish to determine k so that this line shall intersect the circle in two coincident points. To do this, substitute the value of y from (1) in the equation of the circle:

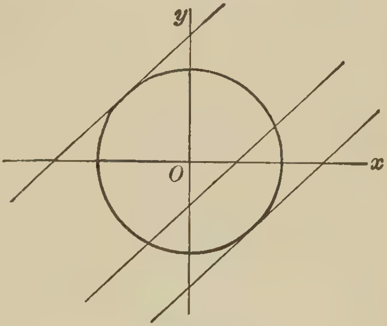


Fig. 43

$$x^2 + (x + k)^2 = 1,$$

or

$$2x^2 + 2kx + k^2 - 1 = 0.$$

The roots of this quadratic in x are the abscissas of the points of intersection of the line (1) with the circle; if these roots are equal the points will coincide. We therefore set $b^2 - 4ac = 0$; i.e.

$$(2k)^2 - 4 \cdot 2(k^2 - 1) = 0.$$

This reduces to

$$-k^2 + 2 = 0,$$

whence

$$k = \pm\sqrt{2},$$

and the required tangents are

$$y = x \pm \sqrt{2}.$$

It may be objected that the above process does not rest squarely upon the definition of tangent, since the tangent here appears as the limiting position of a secant which moves parallel to its original position until the two points of intersection coincide. It is geometrically obvious, however, that in the case of the circle the same limiting position is obtained by this manner of approaching the limit as by the method used in the definition (§ 51), and the same is true for all the curves with which we shall have to deal. An analytic proof of this statement may be written out by a method to be developed in § 98.

EXERCISES

1. Find the tangents to the circle $x^2 + y^2 = 10$ parallel to the line $3x + 2y = 5$.

2. Find tangents to the circle $3x^2 + 3y^2 = 5$ perpendicular to the line $x - 5y = 6$.

3. Find tangents to the circle $2x^2 + 2y^2 = y$ making an angle of 45° with Ox.

4. Find the tangents to the circle $x^2 + y^2 + x - 8y + 6 = 0$ perpendicular to the line $3x + y + 1 = 0$.

5. Find the tangents to the circle $4x^2 + 4y^2 + x + y = 1$ parallel to the line $3x - 2y = 5$.

6. Devise several other methods for finding the tangents to a circle having a given slope, and apply one or more of them to the example of § 53.

7. By two methods, find tangents to the circle $x^2 + y^2 + 2x = 0$ perpendicular to the line $x - 2y = 6$.

8. Find tangents to the circle $x^2 + y^2 + 10x - 6y = 2$ parallel to the line $2x - y = 0$. Solve in two ways.

9. Prove in several ways that the line $4x - 3y = 17$ is tangent to the circle $x^2 + y^2 + 4x = 21$.

10. Determine a so that the circle $x^2 + y^2 + 2ax = 0$ shall touch the line $y = x + 1$. *Ans.* $a = -1 \pm \sqrt{2}$.

11. Determine c so that the circle $x^2 + (y - c)^2 = 1$ shall touch the line $x + 2y = 0$.

12. Find the tangents common to the circles $x^2 + y^2 = 25$, $x^2 + y^2 = 6x + 8y$.

13. Find tangents common to the circles $x^2 + y^2 - 4x - 2y = 10$, $4x^2 + 4y^2 - 24x - 4y - 23 = 0$.

14. For what values of A will the line $Ax + y + 2 = 0$ touch the circle $x^2 + y^2 = 2y$? *Ans.* $A = \pm 2\sqrt{2}$.

15. The y-intercept of a line is twice its x-intercept, and it touches the circle $x^2 + y^2 + 4x = 0$. Find its equation.

16. Find the tangents to the circle $x^2 + y^2 = 20$ at the ends of the diameter of slope 2. Solve in two ways.

17. Prove in two ways that the circles $x^2 + y^2 - 2x - 1 = 0$, $x^2 + y^2 + 6x - 8y + 7 = 0$ are tangent.

18. Prove that the lines

$$y = mx \pm a\sqrt{1 + m^2}$$

are tangent to the circle $x^2 + y^2 = a^2$ for all values of m. Hence show that two tangents may be drawn to any circle parallel to any line.

19. By the method of § 53, find the tangents to the circle $2x^2 + 2y^2 = x$ parallel to the y-axis. Check by solving in another way.

20. Find the equation of a circle with center at (1, 7) tangent to the circle $x^2 + y^2 = 2$. (Two answers.)

21. Find the equation of a circle with center at (1, − 2) touching the circle $x^2 + y^2 = 45$. (Two answers.)

54. Tangent at a given point of contact. The general method of § 52 may be used to find the equation of the tangent to a circle when the point of contact is given. It is simpler, however, to make use of the fact that the tangent is perpendicular to the radius drawn to the point of contact.

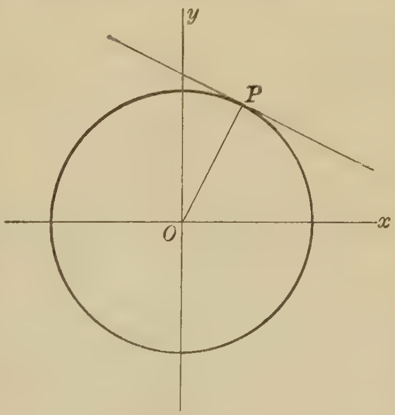


FIG. 44

As an illustration, let us find the tangent to the circle

$$x^2 + y^2 = a^2$$

at any point P: (x_1, y_1) on the curve. The slope of the radius through (x_1, y_1) is evidently $\frac{y_1}{x_1}$; the equation of the required tangent is therefore

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1),$$

or, after clearing of fractions and transposing,

$$(1) \qquad x_1x + y_1y = x_1^2 + y_1^2.$$

But we have not yet introduced the condition that the point (x_1, y_1) is on the curve, viz.

$$x_1^2 + y_1^2 = a^2.$$

Substituting in (1), we obtain the following result:

If the point (x_1, y_1) lies on the circle

$$x^2 + y^2 = a^2,$$

the tangent to the circle at that point is the line

$$(6) \qquad \mathbf{x_1x + y_1y = a^2.}$$

It should be noted carefully that this formula applies *only when the center of the circle is at the origin.*

55. Tangents through an external point. Various methods, based on the known properties of the circle, readily suggest themselves for finding the tangents to a circle through an external point. This problem may also be solved by use of the quadratic discriminant, as in the following

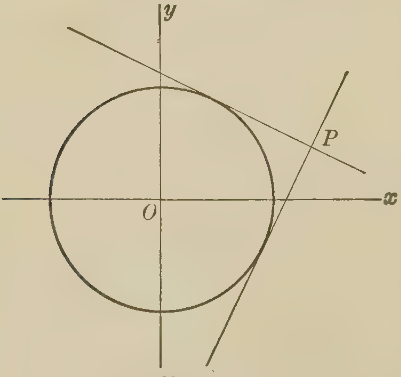


FIG. 45

Example: Find the tangents to the circle

$$(1) \qquad x^2 + y^2 = 5$$

through the point P: (3, 1).

The equation of any line through (3, 1) is

$$(2) \qquad y - 1 = m(x - 3),$$

or

$$y = mx - 3m + 1.$$

Substituting this value of y in (1), we get

$$x^2 + (mx - 3\,m + 1)^2 = 5,$$

which after expanding and collecting terms becomes

$$(1 + m^2)\,x^2 + 2\,m\,(-3\,m + 1)\,x + 9\,m^2 - 6\,m - 4 = 0.$$

This quadratic will have equal roots, and the line (2) will be a tangent, if $b^2 - 4\,ac = 0$: i.e. if

$$4\,m^2\,(-3\,m + 1)^2 - 4\,(1 + m^2)\,(9\,m^2 - 6\,m - 4) = 0.$$

This reduces to

$$2\,m^2 - 3\,m - 2 = 0,$$

or

$$(2\,m + 1)\,(m - 2) = 0.$$

Whence

$$m = -\tfrac{1}{2} \text{ or } 2,$$

and the required tangents are

$$y - 1 = -\tfrac{1}{2}\,(x - 3),$$
$$y - 1 = 2\,(x - 3).$$

EXERCISES

Find the tangents to the following circles at the given points.

1. $x^2 + y^2 = 20$ at $(-2, -4)$.

2. $4\,x^2 + 4\,y^2 = 37$ at $(3, -\tfrac{1}{2})$.

3. $x^2 + y^2 = 5\,x$ at $(1, -2)$. *Ans.* $3\,x + 4\,y + 5 = 0$.

4. $x^2 + y^2 = y$ at $(0, 1)$.

5. $x^2 + y^2 - 4\,x - 6\,y = 0$ at each of the points where it crosses the axes.

6. $x^2 + y^2 - 3\,x - 5\,y = 12$ at the points $x = 1$.

7. $x^2 + y^2 + 2\,x + y = 0$ at the points $x = -1$.
Ans. $y = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}$.

8. Derive formula (6), § 54, by the method of § 52, and then prove analytically that every tangent to a circle is perpendicular to the radius drawn to the point of contact.

9. State and prove the converse of the result of § 54.

10. Devise several other methods for finding tangents through an external point.

11. Solve the example of § 55 by another method.

Find the tangents to the following circles through the given points, using various methods.

12. $x^2 + y^2 = 20$ through $(-6, -2)$.

13. $x^2 + y^2 = 1$ through $(3, 1)$.

14. $3x^2 + 3y^2 + 5x - y + 2 = 0$ through the origin.

15. $2x^2 + 2y^2 + 8y = 1$ through $(0, 4)$.

16. $x^2 + y^2 = 2x$ through $(0, 6)$.

17. $16x^2 + 16y^2 - 40x - 32y + 37 = 0$ through $(\frac{7}{4}, 0)$.

18. Show how to construct geometrically the common tangents to two circles. Discuss special cases.

Find the common tangents to the following circles.

19. $x^2 + y^2 = 4y,\ x^2 + y^2 = 2x + 3$.
Ans. $y = -2x + 2(1 \pm \sqrt{5})$.

20. $4x^2 + 4y^2 = 1,\ (x + \frac{3}{2})^2 + (y - 2)^2 = 4$.

21. $x^2 + y^2 = 1,\ x^2 + y^2 - 6x - 8y + 21 = 0$.

22. $x^2 + y^2 = a^2,\ (x + 2a)^2 + (y + 2a)^2 = a^2$.

23. Given two points P_1, P_2 on a circle, prove analytically that the distance from P_1 to the tangent at P_2 equals the distance from P_2 to the tangent at P_1. What is this distance? *Ans.* $\dfrac{x_1 x_2 + y_1 y_2 - a^2}{a}$.

24. Find the tangents to the circle $x^2 + y^2 + x - 4y + 2 = 0$ through the point $(-2, 4)$, using the method of § 43.

Find the angles between the following curves at their points of intersection.

25. $4x - y + 2 = 0,\ 2x^2 + 2y^2 + x = 0$. *Ans.* $\phi = \arctan \frac{1}{4}$.

26. $x^2 + y^2 - x + 3y = 4,\ 3x + y = 7$. *Ans.* $\phi = \arctan \frac{4}{7}$.

27. $x^2 + y^2 = 10,\ 2x^2 + 2y^2 + 5y = 5x$. *Ans.* $\phi = \arctan \frac{1}{2}$.

28. $x^2 + y^2 = 1,\ x^2 + y^2 - 4x - 2y + 1 = 0$. *Ans.* $\frac{\pi}{2}$.

29. $x^2 + y^2 + 2x = 1,\ x^2 + y^2 - 2x - 4y = 13$. Check the result.

30. What is the analytic condition that the line $Ax + By + C = 0$ and the circle $(x - h)^2 + (y - k)^2 = a^2$ intersect at right angles?
Ans. $Ah + Bk + C = 0$.

31. When do two circles intersect at right angles?

32. Solve Ex. 28 by a second method.

33. Prove that if a line passes at the distance l from the center of a circle of radius a, the angle between the line and the circle is given by the formula $\cos \phi = \frac{l}{a}$.

34. Check the answers to Exs. 25, 26 by the formula of Ex. 33.

56. Chord of contact. If a point P lies outside a circle, the line joining the points of contact of the tangents to the circle through P is called the *chord of contact* corresponding to P, with reference to the given circle.

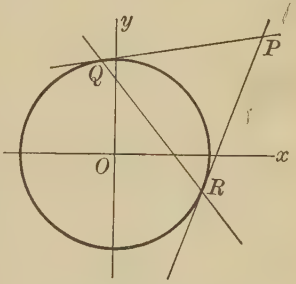


FIG. 46

Given the circle

$$x^2 + y^2 = a^2$$

and an external point P: (x_1, y_1), we proceed to find the equation of the chord of contact corresponding to P. Let the points of contact of the tangents through P (whose coördinates of course are entirely unknown) be denoted by Q: (x_2, y_2) and R: (x_3, y_3). Now the line QP is the tangent to the circle at Q; therefore its equation, by § 54, is

$$x_2x + y_2y = a^2.$$

Similarly the equation of RP is

$$x_3x + y_3y = a^2.$$

By hypothesis, both these lines pass through P; the coördinates of P may therefore be substituted in both equations, and we obtain the relations

(1) $$x_2x_1 + y_2y_1 = a^2,$$

(2) $$x_3x_1 + y_3y_1 = a^2.$$

Consider now the straight line

(3) $$x_1x + y_1y = a^2.$$

This straight line passes through the point Q, since if the coördinates of Q be substituted in (3) for x and y, we obtain the relation

$$x_1x_2 + y_1y_2 = a^2,$$

and this is identical with (1), which is known to be true. Similarly, by (2), the line passes through R. Since the line (3) passes through Q and R, *it is the required chord of contact.*

Thus we have the following result:

If the point $P: (x_1, y_1)$ lies outside the circle

$$x^2 + y^2 = a^2,$$

the equation

$$x_1x + y_1y = a^2 \tag{7}$$

represents the chord of contact corresponding to P.

It must not be forgotten that this formula applies *only when the center of the circle is at the origin.* A convenient method for finding the chord of contact in the general case will be pointed out in § 59.

By means of the chord of contact, a new method may be devised for finding the tangents to a circle through an external point. We need only to get the equation of the chord of contact, find the points of intersection of the chord with the circle, and write the equations of the tangents at those points.

EXERCISES

Write the equation of the chord of contact corresponding to the given point, with reference to the given circle.

1. $x^2 + y^2 = 4$, $(3, 5)$. **2.** $x^2 + y^2 = 1$, $(0, 2)$.

3. $2x^2 + 2y^2 = 75$, $(6, -2)$. **4.** $3x^2 + 3y^2 = 7$, $(-\frac{5}{2}, -\frac{4}{3})$.

5. By a new method, find the tangents to the circle $x^2 + y^2 = 20$ through $(-6, -2)$. (Ex. 12, p. 86.)

6. By a new method, find tangents to the circle $x^2 + y^2 = 1$ through $(3, 1)$. (Ex. 13, p. 86.)

7. Find the tangents to the circle $x^2 + y^2 = 12$ through $(0, 6)$.

8. Find the tangents to the circle $4x^2 + 4y^2 = 9$ through $(-\frac{3}{2}, \frac{5}{2})$.

9. Prove geometrically and analytically that the chord of contact corresponding to a given point is perpendicular to the diameter through that point.

10. Prove that if the chord of contact of a point P with respect to the circle $x^2 + y^2 = a^2$ touches the circle $4x^2 + 4y^2 = a^2$, then P lies on the circle $x^2 + y^2 = 4a^2$.

11. In Fig. 46, draw the line OP intersecting QR at a point K, and prove geometrically that $OP \cdot OK = a^2$.

12. Prove the theorem of Ex. 11 analytically.

13. From the theorem of Ex. 11, devise a geometric construction for the chord of contact corresponding to P, and hence for the tangents to the circle through P.

14. Given two points P_1, P_2 external to a given circle, prove that if the chord of contact of P_1 passes through P_2, then the chord of contact of P_2 passes through P_1.

15. From the theorem of Ex. 14 obtain a geometric construction for the tangents through an external point.

16. Derive formula (7) from the theorems of Exs. 9, 11.

57. Curves through the intersections of two given curves.

Let there be given any two curves, whose equations we will denote by

$$u = 0,\ v = 0, \tag{1}$$

where u and v represent certain unspecified expressions involving x and y. (The student should make sure that he understands the meaning of the abbreviated notation here used. The letters u and v stand for *any expressions in x and y whatsoever*, not necessarily of the first or second degree. The discussion is entirely general.) Suppose these two curves to intersect in certain points P_1, P_2, etc.

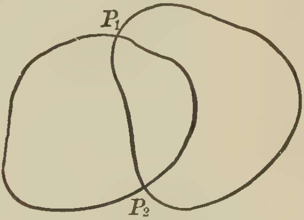


Fig. 47

We now take for consideration the locus

$$u + kv = 0, \tag{2}$$

where k is an arbitrary constant, and proceed to show that, for all values of k, this curve passes through the points of intersection of the two given curves. Since P_1 lies on the curve $u = 0$, its coördinates when substituted for x and y in the expression u, no matter where u is found, will make that quantity identically zero. Similarly, since P_1 lies on the curve $v = 0$, its coördinates will make v vanish. Substituting the coördinates of P_1 in (2), we therefore get

$$0 + k \cdot 0 = 0,$$

which is true for all values of k; hence P_1 lies on the curve (2). By the same reasoning this curve passes through all the other points of intersection of the given curves.

In summary, we have proved the

THEOREM: *If* $u = 0$ *and* $v = 0$ *are two intersecting curves, the equation*

$$u + kv = 0, \tag{8}$$

where k *is an arbitrary constant, represents a family of curves through the points of intersection of the given curves.*

It should be noted that this theorem holds, analytically, even when the original curves intersect only in imaginary points. However, when the equations (1) are incompatible the theorem no longer applies, although in the more advanced theory of curves even this case can be interpreted.

Since equation (8) contains one undetermined constant, the curve may be made to satisfy one condition — for instance to pass through a given point or to touch a given line.

58. Circle through the intersections of two given circles. Let there be given two circles whose equations may be denoted by *

$$C_1 = 0, C_2 = 0.$$

Then the equation

$$C_1 + kC_2 = 0 \tag{1}$$

represents a curve through the points of intersection of the two circles; further, it is easily seen that in general this last curve is also a circle.

---

* That is, the letters C_1, C_2 stand for two expressions of the form $Ax^2 + Ay^2 + Dx + Ey + F$.

The circle (1) is already subject to two conditions, viz. that it must pass through the two points of intersection of the given circles; it may therefore be made to satisfy one more condition. This is also shown, of course, by the fact that (1) contains a single constant, k.

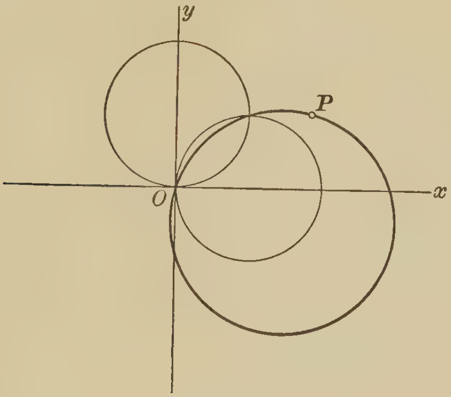


FIG. 48

Example: Find the circle through the points of intersection of the circles

$$x^2 + y^2 = y,\ x^2 + y^2 = x$$

and through the point P: $(1, \frac{1}{2})$.

The required equation may be written in the form

$$x^2 + y^2 - y + k\,(x^2 + y^2 - x) = 0. \tag{2}$$

If this circle is to pass through $(1, \frac{1}{2})$, those coördinates may be substituted for x and y:

$$1 + \tfrac{1}{4} - \tfrac{1}{2} + k\,(1 + \tfrac{1}{4} - 1) = 0,$$

whence

$$k = -3.$$

Substituting this value of k in (2), we get

$$x^2 + y^2 - y - 3\,(x^2 + y^2 - x) = 0$$

or

$$2\,x^2 + 2\,y^2 - 3\,x + y = 0$$

which is the desired result.

EXERCISES

1. Write the equation of the family of straight lines through the point of intersection of the lines $5x + 6y = 1$, $7x - 5y = 2$.

2. In Ex. 1, find that line of the family that passes through $(2, -1)$.

3. In Ex. 1, find the line that is parallel to the y-axis.

4. In Ex. 1, find the line perpendicular to $x - 2y = 6$.

5. Find the line through the intersection of the lines $x - 3y = 6$, $4x + y = 0$ and through the point $(4, 1)$.

6. Find the line through the intersection of $3x + 2y = 1$, $x = 5$ and passing at a distance 2 from the origin.

7. Find the line through the intersection of the lines $x - 5y + 5 = 0$, $2x + y = 3$ and through $(1, 1)$.

8. Find the line through the intersection of $x - 5y = 6$, $2x = 10y + 1$ and through $(4, -3)$. Interpret geometrically.

9. If $C = 0$ represents a circle and $L = 0$ represents a line, what is the locus of the equation $C + kL = 0$?

Find the equations of the following circles.

10. Through the intersections of the circles $x^2 + y^2 = 2x$, $x^2 + y^2 = 2y$ and through $(1, -2)$.

11. Through the intersections of the circles of Ex. 10, with center on the line $y = x$.

12. Through the intersections of the circles $x^2 + y^2 - 4x = 10$, $4x^2 + 4y^2 = 24x + 4y + 23$ and through $(1, 0)$.

13. Through the intersections of the circle $x^2 + y^2 = 25$ and the line $3x + 4y = 1$, and through the point $(3, 2)$.

14. Through the intersections of the circle $x^2 + y^2 - 2x = 1$ and the line $x + 2y = 3$, with center on the y-axis.

15. Through the intersections of the circles $x^2 + y^2 = 20y$, $2x^2 + 2y^2 + x + y = 0$, with center on the x-axis.

16. What is represented by equation (8), § 57, when the original curves are two parallel lines?

17. Prove the statement that equation (1), § 58, represents a circle. Are there any exceptions?

18. Show that the reasoning of § 58 fails when the circles are concentric. What is represented by (1) in that case?

19. Prove geometrically that the center of the circle $C_1 + kC_2 = 0$ is on the line of centers of the circles $C_1 = 0$, $C_2 = 0$. When does the proof fail?

20. Solve Ex. 19 analytically. (For convenience, take the given circles with centers on Ox. Is the proof still general?) What is the advantage of the analytic over the geometric proof?

59. Common chord. Given the equations of two intersecting circles

$$C_1 = 0, C_2 = 0,$$

we can make the coefficients of x^2 and y^2 the same in the two equations if they are not already so. If then we subtract one equation from the other, member by member, the terms of the second degree drop out, so that the resulting equation

$$C_1 - C_2 = 0 \tag{1}$$

is of the first degree and therefore represents a straight line. But since equation (1) is merely that case of (8), § 57, in which $k = -1$, this line passes through the points of intersection of the two circles and is their *common chord.* Hence, to find the equation of the common chord of two intersecting circles we need only to *eliminate x^2 and y^2 between the equations of the circles.*

The above theory gives a convenient method for finding the equation of the chord of contact corresponding to a point P with reference to a given circle. Supposing the center of the circle to be at the point Q, let us write the equation of another circle having the points P, Q as ends of a diameter, and find the common chord of this and the given circle. It is easily seen that this common chord is the required chord of contact.

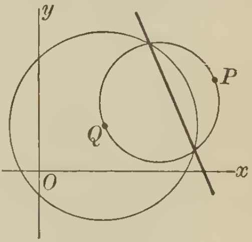


FIG. 49

60. Radical axis. Whether two circles intersect or not, the equation obtained by eliminating the second degree terms represents a real straight line *, and this line is called the *radical axis* of the two circles. The common chord is merely that case of the radical axis in which the circles intersect in real distinct points.

The three radical axes of three circles taken in pairs intersect in a point †, called the *radical center* of the three circles. The proof of this theorem is left to the student.

61. Length of tangent. The *length of the tangent* from an external point P to a circle is defined as the distance from P to the point of contact T. When the point and the circle are given, the distances QT and QP (Fig. 50) are readily determined, after which the length PT may be found at once.

The idea of length of tangent has a number of applications, some of which are brought out in the exercises below.

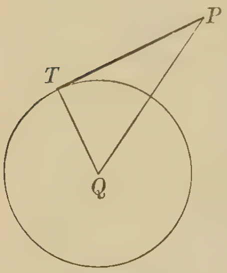


Fig. 50

EXERCISES

1. Find the common chord of the circles $x^2 + y^2 = 10$, $x^2 + y^2 - 8x - 3y = 2$.

2. In Ex. 1, find the length of the chord. *Ans.* $\frac{6}{73}\sqrt{5402}$.

3. Find the common chord of the circles $2x^2 + 2y^2 + 3x + 5y = 9$, $6x^2 + 6y^2 + 11x + 13y = 23$.

4. Find the chord of contact corresponding to the point (3, 5) with reference to the circle $x^2 + y^2 = 2y$; also find the length of the chord. *Ans.* $3x + 4y = 5$; $\frac{4}{5}\sqrt{6}$.

5. Find the chord of contact corresponding to (2, 2), with reference to the circle $x^2 + y^2 + 3x + 2y + 2 = 0$.

6. By a new method, find the tangents to the circle $x^2 + y^2 = 2x$ through (0, 6). (Ex. 16, p. 86.)

---

* There is one exceptional case. In Ex. 13 below the student is asked to discover this case.

† Again there is an exception. See Ex. 17 below.

7. By a new method, find the tangents through $(\frac{7}{4}, 0)$ to the circle $16x^2 + 16y^2 - 40x - 32y + 37 = 0$. (Ex. 17, p. 86.)

8. Find the chord of contact corresponding to the point $(6, -4)$ with reference to the circle $2x^2 + 2y^2 + 2x - 6y + 1 = 0$.

Ans. $13x - 11y + 19 = 0$.

9. Derive (7), § 56, by a new method.

10. What is represented by (1), § 59, when the circles are tangent to each other?

11. Find the radical axis of the circles $x^2 + y^2 = 1$, $(x - 3)^2 + (y - 2)^2 = 4$.

12. Find the radical axis of the circles $x^2 + y^2 = x$, $2x^2 + 2y^2 - 10y + 11 = 0$.

13. In what case do two circles have no radical axis?

14. Prove that the radical axes of three circles taken in pairs, meet in a point.

15. Give a geometric construction for the radical axis of two non-intersecting circles, based on the theorem of Ex. 14.

16. Prove analytically that the radical axis of two circles is perpendicular to their line of centers. Hence vary the construction of Ex. 15.

17. In what case (beside that of concentric circles) do three circles have no radical center?

18. Where is the radical axis of two equal circles situated? Prove analytically.

19. Find the radical center of the circles $x^2 + y^2 = 4$, $x^2 + y^2 = 4x$, $x^2 + y^2 + 8x + 6y + 24 = 0$. Draw the figure.

20. Find the radical center of the circles $2x^2 + 2y^2 = 5$, $x^2 + y^2 - 4x - 8y + 21 = 0$, $3x^2 + 3y^2 + 15x + 12y + 28 = 0$.

21. Find the radical center of the circles $x^2 + y^2 - 6y + 8 = 0$, $x^2 + y^2 = 1$, $2x^2 + 2y^2 + y = 0$. Draw the figure.

22. Find the radical axis of two point circles (i.e. circles of radius 0). Cf. Ex. 18.

23. Solve example (*a*), § 27, by a method based on the result of Ex. 22. Compare the algebraic work of the new method with that of the original method.

24. By a new method, find the perpendicular bisector of the line joining $(-1, -2)$, $(5, -4)$.

25. Prove that the perpendicular bisectors of the sides of a triangle meet in a point.

26. Devise a method for finding the equation of the tangent at a point on a circle by considering the given point as a point circle.

By the method of Ex. 26, find the tangents to the following circles at the given points.

27. $x^2 + y^2 = 2$ at $(-1, 1)$.

28. $2x^2 + 2y^2 - 3x - 4y - 8 = 0$ at $(2, 3)$.

29. $x^2 + y^2 + 2x + 3y = 1$ at $(1, -1)$.

30. $x^2 + y^2 = a^2$ at (x_1, y_1).

31. $Ax^2 + Ay^2 + Dx + Ey + F = 0$ at (x_1, y_1).
Ans. $Ax_1x + Ay_1y + \frac{1}{2}D(x + x_1) + \frac{1}{2}E(y + y_1) + F = 0.$

32. Find the radical center of the circles $x^2 + y^2 = 0$, $x^2 + y^2 - 2x + 2 = 0$, $x^2 + y^2 - 2y + 2 = 0$. Draw the figure.

33. Find the length of the tangent from the point $(3, 3)$ to the circle $x^2 + y^2 = 3$.

34. Find the length of the tangent from $(-1, 0)$ to the circle $x^2 + y^2 - 2x - 6y + 6 = 0$.

35. Find the length of the tangent from $(0, 0)$ to the circle $x^2 + y^2 - 3x + 5y + 8 = 0$.

36. Find the length of the tangent from $(-5, 6)$ to the circle $3x^2 + 3y^2 = 4x + 5y$.

37. If $P: (x, y)$ is a point outside the circle $(x - h)^2 + (y - k)^2 = a^2$, show that the length of the tangent from P to the circle is given by the formula

$$l = \sqrt{(x - h)^2 + (y - k)^2 - a^2}.$$

38. Prove that if the tangents to two circles from an external point are of equal length, the point lies on the radical axis. Hence devise a new construction for the radical axis of two circles. Is this construction general?

39. Using the length of the tangent, develop a new method for finding the chord of contact corresponding to an external point.

40. Solve Ex. 5 by a new method.

41. Find the chord of contact corresponding to the point $(6, -6)$, with reference to the circle $2x^2 + 2y^2 + 8x - 4y + 9 = 0$.

42. Derive formula (7), § 56, by another method.

43. Employ the method of Ex. 39 to find the tangents to the circle $x^2 + y^2 - 6x - 1 = 0$ through (2, 3).

44. Solve Ex. 43 by using the quadratic discriminant.

45. Solve Ex. 43 by a method based on the formula of § 43.

46. By the three methods of Exs. 43–45, find the tangents to the circle $x^2 + y^2 - 4y - 5 = 0$ through (3, 2).

47. A point P moves so that the lengths of the tangents from P to two given circles are in a constant ratio. Prove that the locus of P is a circle through the points of intersection (real or imaginary) of the given circles. ✓

CHAPTER VI

POLAR COÖRDINATES

62. Polar coördinates. Hitherto we have always determined the position of a point by means of its distances from two perpendicular straight lines — i.e. by rectangular cartesian coördinates. Many problems, however, may be much simplified by the use of polar coördinates, which will now be defined.

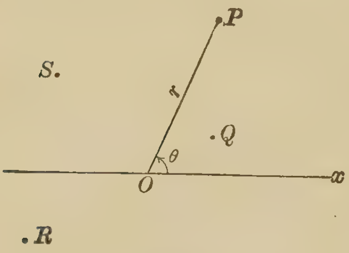


FIG. 51

Let us choose a fixed line Ox in the coördinate plane, and a point O on this line. The position of any point P in the plane is evidently determined if we know the length of the line OP together with the angle that this line makes with the fixed line Ox, both the distance and the angle being measured in a definite sense. The segment OP and the angle xOP are the *polar coördinates* of P; they are called the *radius vector* and the *polar angle* respectively, and are denoted by the letters r, θ. The fixed line Ox is the *initial line*, or *polar axis*, and the point O is the *pole*, or *origin*.

The polar coördinates of a point are written in parenthesis with the radius vector first: thus the point P is indicated as $P: (r, \theta)$, or simply (r, θ). The polar angle is positive when measured counterclockwise, negative clockwise; the radius vector is positive if laid off on the terminal side of θ, negative if measured in the opposite sense, i.e. on the terminal side produced through O. Fig. 51 shows the points Q: $(1, 30°)$, R: $(-2, 30°)$, S: $\left(-2, -\frac{\pi}{4}\right)$.

To plot a point whose polar coördinates are given, it is best to begin by drawing the line on which the radius vector lies, i.e. the line making an angle θ with Ox, and then to lay off on that line, in the proper sense, the distance r. This is done most conveniently on "polar coördinate paper," which is paper ruled in concentric circles and radial lines.

While to every pair of polar coördinates corresponds a single definite point, the converse is not true: the same point may be represented by various pairs of coördinates. For instance, in Fig. 51, the coördinates $(1, 30°)$, $(-1, 210°)$, $(-1, -150°)$, $(1, 390°)$, all represent the same point Q.

63. Curve tracing in polar coördinates. If r and θ are connected by an equation of any form, we may assign values to θ and compute the corresponding value or values of r. The points thus determined all lie on a definite curve, the *locus* of the equation.

To each equation corresponds a single definite curve, but while in cartesian coördinates there is only one equation corresponding to a given curve, this is not true in the polar system: the fact that a given point may be represented by different pairs of coördinates makes it possible that a curve may be represented by more than one equation. Thus it is easily seen that the equations $r = 2$, $r = -2$ represent exactly the same curve, viz. a circle of radius 2 with center at O.

Example: Trace the curve

$$r = 2a \cos \theta.$$

Let us assign values to θ and compute r:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	$2a$	$\sqrt{3}a$	$\sqrt{2}a$	a	0	$-a$	$-\sqrt{2}a$	$-\sqrt{3}a$	$-2a$

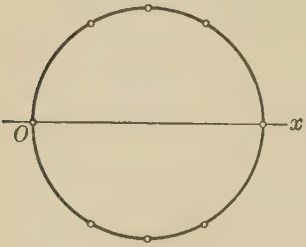


FIG. 52

Plotting these points, we obtain the curve shown in Fig. 52, which appears to be a circle of radius a with center at $(a, 0)$; it will be shown in the next article that this is actually the case. It is easily seen from the properties of the cosine function that if we were to assign values to θ between π and 2π (or between 0 and $-\pi$) we would merely get the same circle over again; the complete locus is therefore as shown in the figure.

EXERCISES

Plot the following points.

1. $(2, 60°)$, $(3, \pi)$, $(-4, -15°)$, $(0, 20°)$, $\left(4, -\frac{\pi}{2}\right)$, $\left(-4, \frac{\pi}{2}\right)$.

2. $\left(\frac{a}{3}, \frac{\pi}{3}\right)$, $\left(\frac{a}{10}, 10°\right)$, $(\frac{1}{3}\sqrt{3}\,a,\ 225°)$, $(-\frac{1}{2}\sqrt{2}\,a,\ -60°)$.

Trace the following curves.

3. $r = a$.

4. $r = -a$.

5. $\theta = 60°$.

6. $\theta = -\frac{\pi}{4}$.

7. $r = 2\sin\theta$.

8. $r = -2\sin\theta$.

9. Prove that a curve is symmetric with respect to the initial line if its equation is unchanged when θ is replaced by $-\theta$.

10. What can be said of the symmetry of a curve whose equation is unchanged when r is replaced by $-r$?

11. What can be said of the symmetry of a curve whose equation is unchanged when θ is replaced by $\pi - \theta$?

12. What can be said of the symmetry of a curve whose equation is unchanged when the signs of θ and r are changed simultaneously?

13. What can be said of the symmetry of a curve whose equation is unchanged when θ is replaced by $\pi - \theta$ and r by $-r$?

Test the following curves for symmetry with respect to the polar axis, the pole, and the line perpendicular to the axis through the pole, plot a number of points, and trace the curves.

14. $r = 3\sec\theta$.

15. $r\sin\theta = a$.

16. $r = a \cos 2\theta.$

17. $r = \dfrac{1}{1 - \cos\theta}.$

18. $r = \dfrac{2a}{1 - \sin\theta}.$

19. $r = a \sin 2\theta.$

20. $r^2 = a^2 \cos\theta.$

21. $r^2 = a^2 \sin 2\theta.$

22. $r = \dfrac{a}{2 - \sin\theta}.$

23. $r^2 \sin 2\theta = a^2.$

24. $r = a\theta.$

25. $r = a(1 + \cos\theta).$

Find the points of intersection of the following curves. Check the results.

26. $r \sin\theta = 1,\ r\cos\theta = 2.$ *Ans.* $(\sqrt{5},\ \arctan \tfrac{1}{2}).$

27. $r^2 \cos 2\theta = a^2,\ r = \sqrt{2}\,a.$

28. $r = 4 \sec\theta \tan\theta,\quad r\sin\theta = 4.$

29. $r(1 - \cos\theta) = a,\ r = a(1 + \cos\theta).$

30. $r = a\cos 2\theta,\ r = a\sin\theta.$

64. Locus problems in polar coördinates. There are various loci whose equations are more easily obtained or whose properties are more conveniently studied in polar than in cartesian coördinates. The methods used in finding the equation of a locus in polar coördinates are of course the same as those that were used in §§ 27, 28. That is, we assume a point $P: (r, \theta)$ in a general position on the curve, and from the statement of the problem or by means of some characteristic property of the figure try to obtain an equation involving r, θ, and the constants of the problem: this equation must be the equation of the given locus.

Example: Find the equation of a circle of radius a with center at $(a, 0)$.

Let us assume a point $P: (r, \theta)$ in a general position on the circle. Since $\angle OPA$ is a right angle, it follows at once that

$$\cos\theta = \frac{r}{2a},$$

or

$$r = 2a\cos\theta.$$

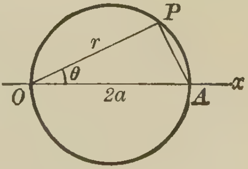


FIG. 53

EXERCISES

Find the polar equations of the following loci. Draw the figures.

1. A circle of radius 5 with center at O.

2. The initial line.

3. A line through O making a given angle α with Ox.

4. The path of a point which moves so that the projection of its radius vector upon the initial line is constant.

5. A circle of radius a touching the initial line at O and lying (a) above, (b) below that line.

6. A line through $(a, 0)$ perpendicular to the initial line.

7. A line through $\left(a, \frac{\pi}{2}\right)$ parallel to the initial line.

8. A line through $\left(2, \frac{\pi}{6}\right)$ perpendicular to the radius vector of that point.

9. A circle of radius a with center at $\left(a, \frac{\pi}{4}\right)$.

10. A line through the point $(1, \frac{3}{4}\pi)$ perpendicular to the radius vector of that point.

11. The path of a point which moves so as to remain always equidistant from a given point and a given line. Take the given point as pole and the given line perpendicular to the polar axis through $(a, 0)$.

Ans. $r = \dfrac{a}{1 + \cos\theta}$.

12. Solve Ex. 11, taking the given line parallel to Ox through $\left(-a, \frac{\pi}{2}\right)$.

Ans. $r = \dfrac{a}{1 - \sin\theta}$.

13. Solve Ex. 11, taking the line through the point $\left(a, \frac{\pi}{6}\right)$ perpendicular to the radius vector of that point.

14. Solve Ex. 11, if the moving point is half as far from the given point as from the given line.

15. Solve Ex. 11, taking the given line through the point (r_1, θ_1) perpendicular to the radius vector of that point.

16. Solve the problem of § 39 in polar coördinates.

65. Transformation from one system to the other. If the pole and the initial line of a polar system coincide with the origin and the x-axis of a cartesian system, as is usually the case, the coördinates of any point P in the two systems are connected by simple formulas.

Let the point P (Fig. 54) have the cartesian coördinates x, y and the polar coördinates r, θ. Then it is obvious that

$$(1)\qquad \begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases}$$

and

$$(2)\qquad \begin{cases} r = \sqrt{x^2+y^2}, \\ \tan\theta = \dfrac{y}{x}. \end{cases}$$

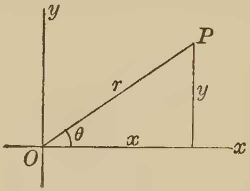


FIG. 54

These formulas enable us to change the equation of a curve from cartesian coördinates to polar and from polar to cartesian respectively.

Example: Obtain the equation of the circle $x^2 + y^2 = 2\,ax$ in polar coördinates.

Substituting the values of x and y as given by (1), we find

$$r^2\cos^2\theta + r^2\sin^2\theta = 2\,ar\cos\theta,$$

or, since

$$\cos^2\theta + \sin^2\theta = 1,$$

$$r = 2\,a\cos\theta.$$

The result might have been obtained more quickly by noting that $x^2 + y^2$ can be replaced by r^2 at once.

EXERCISES

In the following exercises it is assumed that the axes are placed as in Fig. 54.

1. Find the cartesian coördinates of the points (*a*) $(1, 30°)$; (*b*) $(-1, \frac{1}{4}\pi)$; (*c*) $(-2, 0)$; (*d*) $(3, -60°)$; (*e*) $(-2, \frac{5}{6}\pi)$; (*f*) $(3, 225°)$. Check by plotting.

2. Find the polar coördinates of the points (*a*) $(2, 2)$; (*b*) $(\sqrt{3}, -1)$; (*c*) $(0, 0)$; (*d*) $(-3, -4)$; (*e*) $(-5, -5)$; (*f*) $(-1, \frac{1}{3}\sqrt{3})$. Check by plotting.

Find the equations of the following curves in polar coördinates.

3. $y = x$.

4. $x^2 + y^2 = a^2$.

5. $x = a$.

6. $x^2 + y^2 + y = 0$.

7. $3x + 4y = 5$.

8. $y = x^2$.

9. $2xy = a^2$.

10. $x^2 - y^2 = a^2$.

Find the equations of the following curves in cartesian coördinates.

11. $r = a$.

12. $\theta = 45°$.

13. $r \sin \theta = 1$.

14. $r = 2 \sin \theta$.

15. $r = \sec \theta \tan \theta$.

16. $r^2 \cos 2\theta = a^2$.

17. $r^2 = a^2 \sin 2\theta$.

18. $r = a(1 - \cos \theta)$.

19. Solve Ex. 11, p. 102, in cartesian coördinates and check by transforming the answer $r = \frac{a}{1 + \cos \theta}$ to cartesian coördinates.

20. Solve Ex. 12, p. 102, in cartesian coördinates and check by transforming one answer to the other.

CHAPTER VII

THE CONIC SECTIONS

I. General Introduction

66. Definitions. The path of a point which moves so that *its distance from a fixed point is in a constant ratio to its distance from a fixed line* is called a *conic section,* or simply a *conic.* The name is taken from the fact, proved in solid analytic geometry, that the intersection of a plane and a right circular cone is always a curve of the class here defined.

The fixed point is called the *focus* of the conic, the fixed line the *directrix,* and the constant ratio the *eccentricity.* In Fig. 55, if F is the focus, λ the directrix, and P a point on the conic, then the definition evidently says that

$$\frac{FP}{LP} = e,$$

or

$$\rho = ed, \tag{1}$$

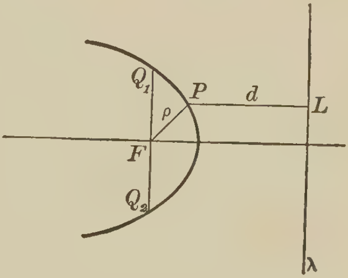


Fig. 55

where e denotes the eccentricity.

It is geometrically obvious that the line through the focus perpendicular to the directrix is an axis of symmetry for the curve. It is also easily seen that the line through the focus parallel to the directrix intersects the curve in two points; the chord Q_1Q_2 joining these two points is called the *latus rectum.*

67. Species of conics. We shall find that the conic sections fall into three classes, differing greatly in form and in certain of their properties, and distinguished by the value of e. The classification is as follows:

if $e < 1$, the conic is an *ellipse;*
if $e = 1$, the conic is a *parabola*;
if $e > 1$, the conic is a *hyperbola.*

It will appear as we proceed that the ellipse has the form of a closed oval, while the parabola is an open curve extending to infinity, and the hyperbola consists of two separate infinite branches. A curve of each species is shown in Fig. 56.

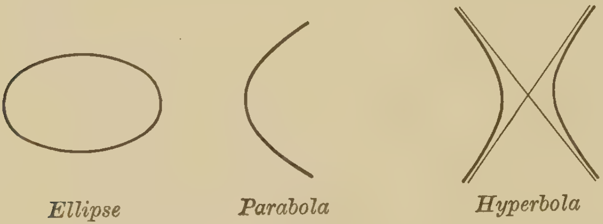


FIG. 56

In the present chapter we will solve for the conic sections the first two fundamental problems of analytic geometry,* viz.: to find the equation of the curve from its definition, and to trace the curve from its equation; in the next two chapters we will develop the properties of the three curves.

68. General equation. Let the focus of a conic be the point (i, j), and the directrix the line

$$x \cos \beta + y \sin \beta = p. \tag{1}$$

By § 41, the distance of the moving point (x, y) from the line (1) is $x \cos \beta + y \sin \beta - p$. Hence, by (**1**), § 66, the equation of the conic is

$$\sqrt{(x-i)^2 + (y-j)^2} = e\,(x \cos \beta + y \sin \beta - p),$$

or

$$(x-i)^2 + (y-j)^2 = e^2\,(x \cos \beta + y \sin \beta - p)^2.$$

---

* See § 29.

This is an equation of the second degree; hence we have proved the important

THEOREM: *The equation of every conic is of the second degree.*

The converse of this theorem is also true, as we shall prove in § 86.

69. Polar equation of a conic. To find the equation of a conic in polar coördinates, let us take the focus as pole and the line through the focus perpendicular to the directrix as polar axis. Denote by $2l$ the length of the latus rectum Q_1Q_2 (Fig. 57); then the distance OB from the focus to the directrix λ is $\frac{l}{e}\cdot$ For, the point Q_1 is a point on the curve; hence, by the definition of § 66,

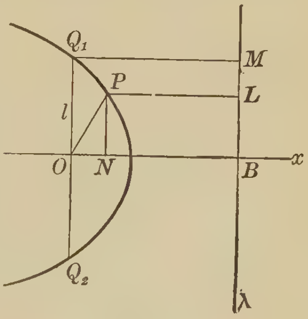


FIG. 57

$$OQ_1 = e\cdot Q_1M,$$

so that

$$OB = Q_1M = \frac{OQ_1}{e} = \frac{l}{e}\cdot$$

Now assume a point P: (r, θ) in a general position on the curve, and drop a perpendicular PL to the directrix. Since

$$OP = e\cdot PL,$$

it follows that

$$PL = \frac{r}{e}\cdot$$

Evidently

$$ON + PL = OB,$$

or

$$r\cos\theta + \frac{r}{e} = \frac{l}{e}\cdot$$

This equation when solved for r takes the form

(2) $$r = \frac{l}{1 + e\cos\theta}\cdot$$

EXERCISES

Derive the cartesian equations of the following conics.

1. Focus at (0, 0), directrix $x = 2$, $e = 1$. *Ans.* $y^2 = -4x + 4$.

2. Focus at (0, 0), directrix $y = 3$, $e = \frac{1}{3}$.
Ans. $9x^2 + 8y^2 + 6y = 9$.

3. Focus at (1, 0), directrix $x + 4 = 0$, $e = \sqrt{2}$.
Ans. $x^2 - y^2 + 18x + 31 = 0$.

4. Focus at (3, 2), directrix $x + y = 0$, $e = 1$.
Ans. $x^2 - 2xy + y^2 - 12x - 8y + 26 = 0$.

5. Focus at $(-1, -3)$, directrix $2x - y = 3$, $e = \frac{2}{3}$.

6. Focus at $(-6, 0)$, directrix $3x - 2y + 1 = 0$, $e = 3$.

7. A point moves so that its distance from the line $3x - y = 6$ is always twice its distance from the point (2, 4). Find the equation of its path. What kind of curve is described?

8. Prove that if the directrix of a conic is parallel to a coördinate axis the equation contains no xy-term.

Derive the equations of the following conics in polar coördinates, taking the focus as pole in each case.

9. With $e = 1$ and directrix perpendicular to Ox through $(3, \pi)$.
Ans. $r = \dfrac{3}{1 - \cos\theta}$.

10. With $e = \frac{1}{2}$ and directrix perpendicular to Ox through (2, 0).
Ans. $r = \dfrac{2}{2 + \cos\theta}$.

11. With $e = 4$ and directrix parallel to Ox through the point $\left(3, \frac{\pi}{2}\right)$.
Ans. $r = \dfrac{12}{1 + 4\sin\theta}$.

12. With $e = \frac{4}{5}$ and directrix parallel to Ox through $\left(2, \frac{\pi}{2}\right)$.

13. With $e = 1$ and directrix parallel to Ox through $\left(5, -\frac{\pi}{2}\right)$.

14. With $e = \frac{3}{2}$ and directrix perpendicular to Ox through $(-3, 0)$.

15. Trace the conic in Ex. 9.

16. Trace the conic in Ex. 10.

17. Classify each of the conics in Exs. 9–14.

18. Compare the loci $r = \frac{l}{1 + e \cos \theta}$ and $r = \frac{-l}{1 - e \cos \theta}$.

19. Prove analytically that the latus rectum intersects the conic in two points.

20. Prove analytically that a conic is symmetric with respect to the line through the focus perpendicular to the directrix. Solve in two ways: (*a*) using cartesian, (*b*) using polar coördinates.

21. Prove that the line through the focus perpendicular to the directrix intersects an ellipse or hyperbola in two points, but a parabola in only one point.

22. Prove that a straight line can intersect a conic in not more than two points.

23. How many points are required to determine a conic?

24. How many points are required to determine a parabola?

How many points are required to determine a conic:

25. If the focus is given?

26. If the directrix is given?

27. If the eccentricity is given?

28. If the focus and directrix are given?

29. If the direction of the directrix is given?

30. If the distance from the focus to the directrix is given?

II. THE PARABOLA

70. Definitions. The parabola has been defined in § 67 as the conic section whose eccentricity is 1; or in other words: *The parabola is the locus of points which are equidistant from a fixed point and a fixed line.*

From this definition any number of points on the curve may be constructed geometrically if the focus and directrix are known.

The line through the focus perpendicular to the directrix is called the *axis* of the curve. It has been pointed out in § 66, and appears also from the above-mentioned geometric construction, that a parabola is *symmetric with respect to its axis.*

The point where the axis intersects the curve, i.e. the point midway between the focus and the directrix, is the *vertex* of the parabola. The distance from the vertex to the focus will be denoted by the letter a.

71. Parabola with vertex as origin: first standard form. Let us take the vertex of a parabola as origin of coördinates and the focus at the point $(a, 0)$. Then the axis of the curve is the x-axis and the directrix is the line $x = -a$. If $P: (x, y)$ is any point on the curve, then by the definition of the parabola

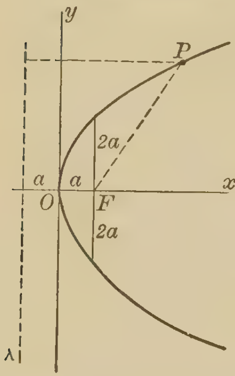


FIG. 58

$$\sqrt{(x-a)^2 + y^2} = a + x,$$

or

$$x^2 - 2\,ax + a^2 + y^2 = a^2 + 2\,ax + x^2,$$

which reduces to

$$(3) \qquad \mathbf{y^2 = 4\,ax.}$$

This is the *first standard form* of the equation of the parabola; on account of its simplicity it is the equation that will be most used in developing the properties of the curve.

From equation (3) it appears that a and x must have the same sign, otherwise their product would be negative and y would be imaginary; hence the curve opens to the right or to the left according as a is positive or negative. For every value of x there are two values of y, numerically equal but of opposite sign, which grow steadily larger as x grows larger: the curve is an open curve extending to infinity, as shown in Fig. 58.

When $x = a$, $y = \pm 2\,a$, so that the ends of the latus rectum are the points $(a, 2\,a)$, $(a, -2\,a)$. Hence *the length of the latus rectum is four times the distance between the vertex and focus.*

The equation

$$x^2 = 4\,ay$$

evidently represents a parabola with vertex at O and axis in Oy.

72. Other standard forms. The equation

$$y^2 = 4\,ax \tag{1}$$

expresses a characteristic geometric property of the parabola — a property true for all parabolas, regardless of their position with reference to the coordinate axes. Given a parabola with vertex V and focus F (Fig. 59), let us drop from any point P of the curve a perpendicular MP to the axis. Then we have at once, by (1),

$$MP^2 = 4\,VF\cdot VM; \tag{2}$$

this is the property just referred to.

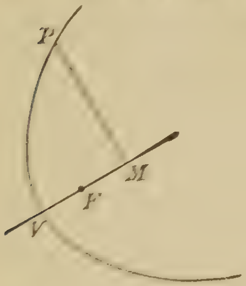


Fig. 59

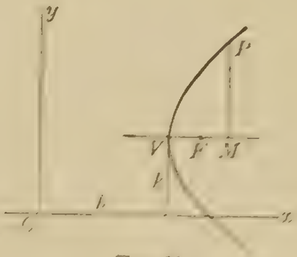


Fig. 60

Consider now a parabola with vertex at (h, k) (Fig. 60), axis parallel to Ox, and focus at a distance a from the vertex. Choosing a point $P:(x, y)$ on the curve, we see at once by (2) that the equation of the parabola is

$$(y - k)^2 = 4\,a\,(x - h).$$

We may proceed similarly when the axis is parallel to Oy.

In summary, we have the following result:

The equation of a parabola with vertex at (h, k) is, if the axis is parallel to Ox,

$$(y - k)^2 = 4\,a\,(x - h); \tag{4}$$

if the axis is parallel to Oy,

$$(x - h)^2 = 4\,a\,(y - k). \tag{5}$$

The curve opens in the positive or the negative direction according as a is positive or negative.

73. General equation. An equation of the form

$$Cy^2 + Dx + Ey + F = 0 \tag{6}$$

can evidently be reduced to the form

$$(y - k)^2 = 4\,a\,(x - h)$$

by dividing by C and completing the square in y; similarly the equation

$$Ax^2 + Dx + Ey + F = 0 \tag{7}$$

can be reduced to the form

$$(x - h)^2 = 4\,a\,(y - k).$$

Exception arises when, in (6), $D = 0$ or when, in (7), $E = 0$, in which cases the given equation represents a pair of parallel straight lines. If we agree to consider the pair of parallel lines as a limiting form of the parabola *, we have the

THEOREM: *An equation of the second degree in which the xy-term is missing and only one square term is present represents a parabola with its axis parallel to a coördinate axis.*

Example: Reduce the equation

$$4\,y^2 - 24\,x - 12\,y - 15 = 0$$

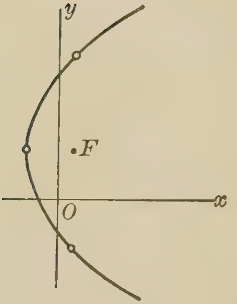


FIG. 61

to a standard form, plot the vertex, focus, and ends of the latus rectum, and trace the curve.

First we divide by 4:

$$y^2 - 6\,x - 3\,y - \tfrac{15}{4} = 0.$$

Transposing the term in x and the constant, and completing the square in y, we find

$$y^2 - 3\,y + \tfrac{9}{4} = 6\,x + \tfrac{15}{4} + \tfrac{9}{4},$$

or

$$(y - \tfrac{3}{2})^2 = 6\,(x + 1).$$

Hence the vertex is at $(-1, \tfrac{3}{2})$, the curve opens to the right, with $a = \tfrac{3}{2}$, a number of points are easily plotted, and the curve is drawn as shown in the figure.

---

* In § 90 it will be shown how the pair of parallel lines may be thought of as satisfying the definition of the parabola.

EXERCISES

In each of the following cases, locate the vertex, the ends of the latus rectum, and a few other points, and trace the curve neatly on a suitable scale.

1. $y^2 = 4x$. **2.** $y^2 = -\frac{3}{2}x$.

3. $y^2 = 36x$. **4.** $x^2 = 8y$.

5. $y + 100x^2 = 0$. **6.** $x^2 + 5y = 0$.

7. $y^2 = 3(x + 2)$. **8.** $(y + 2)^2 = -6(x - 1)$.

9. $(x - \frac{1}{2})^2 = -(y + \frac{3}{4})$. **10.** $(y - 5)^2 = 120x$.

11. $(x + 6)^2 = 2(y - 1)$. **12.** $(x - 0.1)^2 = -0.2(y + 0.2)$.

13. Find the focus, directrix, and ends of the latus rectum of the parabola $(y - k)^2 = 4a(x - h)$.

14. Prove the statement in the last sentence of § 72.

15. As a point on a parabola recedes to infinity, does its distance from the axis increase more or less rapidly than its distance from the tangent at the vertex?

Reduce the following equations to a standard form, determine the vertex, focus, and latus rectum, and trace the curve.

16. $y^2 + 4x + 6y + 9 = 0$. **17.** $3y^2 + 2x - 3y = 0$.

18. $x^2 - 2x - 3y + 4 = 0$. **19.** $x^2 + 8y + 10 = 0$.

20. $5x^2 - 5x + 4y + 3 = 0$. **21.** $2y^2 - 7x - 9y = 0$.

22. $x^2 - 20x + y = 10$. **23.** $3y^2 - 5x + 7y + 2 = 0$.

24. $6y^2 - x - y + 1 = 0$. **25.** $y^2 - 4y + 3 = 0$.

26. $9x^2 + 6x + 1 = 0$. **27.** $y^2 - 150x + 200y = 0$.

28. Show that when $D = 0$, equation (6) represents a pair of parallel lines. Are these lines necessarily real? Distinct?

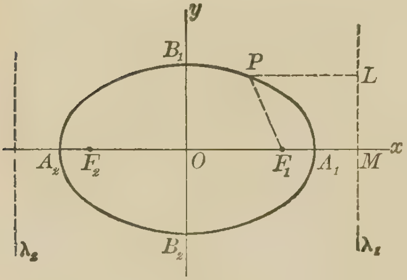
29. Discuss the locus $Ax^2 + Dx + F = 0$.

30. What locus is obtained in the special case when the focus of a parabola lies on the directrix? Establish the result both geometrically and analytically.

III. The Ellipse

74. Ellipse referred to its axes: first standard form. The ellipse has been defined in § 67 as the conic section for which $e < 1$.

To obtain the cartesian equation of the curve, let us take as focus the point $(ae, 0)$, and as directrix the line $x = \frac{a}{e}$ (the point F_1 and line λ_1 of Fig. 62), where a is a constant whose geometric meaning is as yet undetermined. With these assumptions the defining equation (1) of § 66, viz.



$$\rho = ed,$$

becomes

$$\sqrt{(x - ae)^2 + y^2} = e\left(\frac{a}{e} - x\right),$$

or

$$x^2 - 2\,aex + a^2e^2 + y^2 = a^2 - 2\,aex + e^2x^2,$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

For simplicity, let us put *

$$(8) \qquad b^2 = a^2(1 - e^2),$$

thus reducing equation (1) to the form

$$(9) \qquad \frac{x^2}{a_2} + \frac{y^2}{b^2} = 1.$$

This is the simplest standard form of the equation of the ellipse, and is the most convenient form for studying the properties of the curve.

---

* Since $e < 1$, b^2 is positive and b is real.

From equation (9) the following results are evident:

(*a*) The curve is symmetric with respect to Ox (already noted in § 66) and also with respect to Oy. From the latter statement it follows that there is *a second focus, the point* $(-ae, 0)$, and *a second directrix, the line* $x = -\frac{a}{e}$. The curve is also symmetric with respect to the origin.

(*b*) The intercepts on the axes are $(\pm a, 0)$, $(0, \pm b)$. Thus the constant a appears in the figure as the segment OA_1.

(*c*) The curve lies entirely within the rectangle bounded by the lines $x = \pm a$, $y = \pm b$. For the equation when solved for y has the form

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}, \tag{2}$$

which shows that y is imaginary when x is numerically greater than a; similarly, by solving for x, we see that x is imaginary when y is greater than b.

(*d*) *The latus rectum is* $\frac{2b^2}{a}$. For, by (2), when

$$x = ae,$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - a^2e^2} = \pm b\sqrt{1 - e^2} = \pm \frac{b^2}{a}.$$

It is convenient occasionally to speak of the two lines of symmetry as the *axes* of the curve, but except where the contrary is clearly indicated we shall consider the axes to be the *segments* of these lines included within the curve: the segment A_2A_1 is the *transverse axis*, the segment B_2B_1 the *conjugate axis*. The extremities of the transverse axis are the *vertices*, and the point of intersection of the axes is the *center*.

It should be noted that, for the ellipse, b *is always less than* a, since b is given by the formula

$$b^2 = a^2 - a^2e^2.$$

For this reason the transverse and the conjugate axis are often called the *major* and the *minor* axis respectively.

75. Other standard forms. The equation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \tag{10}$$

represents an ellipse with center at (h, k) and major axis parallel to Ox. This can be shown either directly from the definition, or by a method analogous to that used in § 72.

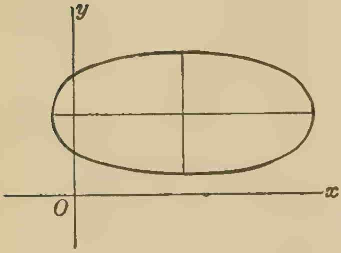


FIG. 63

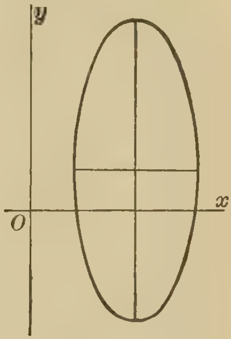


FIG. 64

Similarly, the equation

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1 \tag{11}$$

represents an ellipse with its center at (h, k) and major axis parallel to Oy.

When the equation of an ellipse is given in a standard form, the center and the ends of the axes may be plotted immediately. The foci are most easily found from the fact that, by (8), § 74,

$$ae=\sqrt{a^2-b^2}, \tag{12}$$

so that the foci are at the distance $\sqrt{a^2-b^2}$ from the center. The ends of the latera recta are then easily plotted; this gives a total of eight points on the curve, from which a fairly accurate sketch can be made without the determination of any other points.

76. General equation. It is easily seen that, by completing the squares in x and y, an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{13}$$

can be reduced to one of the forms (**10**), (**11**) of § 75, provided A and C have the same sign. Thus we have the

THEOREM: *An equation of the second degree in which the xy-term is missing and the coefficients of x^2 and y^2 have the same sign represents an ellipse with its axes parallel to the coördinate axes.*

Limiting cases may arise similar to those discussed in § 47. Thus when the left member has been written as the sum of two squares the right member may be zero or negative, the corresponding locus being a so-called point-ellipse or an imaginary ellipse. These limiting forms are of little importance.

Example: Reduce the equation

$$9x^2 + 4y^2 - 36x + 8y + 31 = 0$$

to a standard form, and trace the curve.

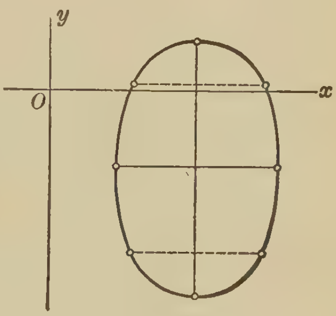


FIG. 65

Transposing the constant term and completing the squares, we find

$$9(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = -31 + 36 + 4,$$

$$9(x-2)^2 + 4(y+1)^2 = 9,$$

or after dividing by 9,

$$\frac{(x-2)^2}{1} + \frac{(y+1)^2}{\frac{9}{4}} = 1.$$

The center is at $(2, -1)$, the transverse axis is parallel to Oy, the semi-axes are $a = \frac{3}{2}$, $b = 1$, the distance from the center to the foci is $\sqrt{a^2 - b^2} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$, the latus rectum is $\frac{2b^2}{a} = \frac{4}{3}$. The curve is shown in Fig. 65.

EXERCISES

1. State explicitly the reasons for the statement (*a*) of § 74.

2. Prove geometrically, from the definition of the ellipse, that the latus rectum is $\frac{2b^2}{a}$.

In each of the following cases, find the center and foci, plot the ends of the axes and of the latera recta, and make a careful sketch of the curve.

3. $\frac{x^2}{25}+\frac{y^2}{9}=1.$

4. $\frac{x^2}{9}+\frac{y^2}{8}=1.$

5. $\frac{x^2}{4}+\frac{y^2}{16}=1.$

6. $12x^2+y^2=12.$

7. $\frac{(x-2)^2}{1}+\frac{y^2}{3}=1.$

8. $\frac{(x+\frac{1}{2})^2}{\frac{1}{4}}+\frac{(y-\frac{1}{4})^2}{1}=1.$

9. $\frac{(x-6)^2}{100}+\frac{(y+7)^2}{75}=1.$

10. $\frac{(x+2)^2}{100}+\frac{(y-3)^2}{1}=1.$

11. In Exs. 3–6, find the eccentricity and the equations of the directrices.

Reduce each of the following equations to a standard form, and trace the curve.

12. $x^2+4y^2=4y.$

13. $2x^2+y^2+4x+2y+1=0.$

14. $15x^2+16y^2-90x+64y-41=0.$

15. $100y^2=20x-x^2.$

16. $4x^2+2y^2-12x-8y+13=0.$

17. $800x^2+400y^2-320x+40y+29=0.$

18. $3x^2+7y^2-12x+28y+19=0.$

19. $5x^2+5y^2+3x-6y=0.$

20. $6x^2+5y^2+8x-2=0.$

IV. The Hyperbola

77. Hyperbola referred to its axes: first standard form. By § 67, the hyperbola is the conic section for which $e > 1$.

The derivation of equation (1), § 74, does not depend upon the fact that $e < 1$; we therefore conclude at once that, with the axes chosen as in § 74, the equation of the hyperbola is

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

But, since now $e > 1$, the constant $a^2(1 - e^2)$ is negative; therefore to make b real, we choose

$$(14) \qquad \mathbf{b^2 = a^2(e^2 - 1),}$$

and write the equation of the hyperbola in the form

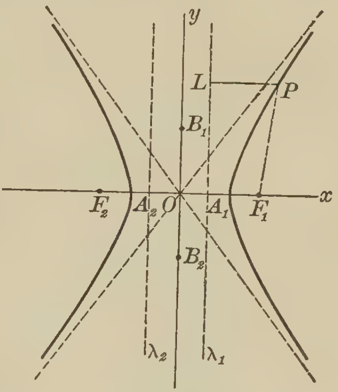


Fig. 66

$$(15) \qquad \boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

From this equation we derive the following results:

(*a*) The curve is symmetric with respect to both axes and the origin. Hence there are two foci, the points $(\pm ae, 0)$, and two directrices, the lines $x = \pm \frac{a}{e}\cdot$

(*b*) The intercepts on Ox are $(\pm a, 0)$, those on Oy are imaginary.

(*c*) The equation when solved for y has the form

$$(2) \qquad y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

which shows that y is imaginary if and only if $x^2 < a^2$, i.e. if $-a < x < a$. The curve therefore has two disconnected branches, one lying to the right of the line $x = a$, the other to the left of the line $x = -a$.

(*d*) *The latus rectum is* $\boldsymbol{\frac{2b^2}{a}}\cdot$

As in the case of the ellipse, the lines of symmetry are sometimes spoken of as the *axes* of the curve, but unless the contrary is indicated the axes will be considered to be the *segments* A_2A_1 of length $2a$ and B_2B_1 of length $2b$: the former is the *transverse axis*, the latter the *conjugate axis*. (The conjugate axis does not intersect the curve, but plays an important part in its theory.) The ends A_2, A_1 of the transverse axis are the *vertices;* the point of intersection of the axes is the *center*.

It appears from (**14**) that the distance from the center to the foci is

$$ae = \sqrt{a^2 + b^2}. \tag{16}$$

It should also be noted that in the case of the hyperbola b may be greater than, equal to, or less than a, according to the value of e.

78. Asymptotes. By dividing both members by x, equation (2) of § 77 may be put in the form

$$\frac{y}{x} = \pm \frac{b}{a}\sqrt{1 - \frac{a^2}{x^2}}.$$

If now x be allowed to increase indefinitely, the right member approaches $\pm \frac{b}{a}$. Hence, as the distance from the center increases, the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

approaches more and more closely, without ever reaching them, the straight lines

$$\frac{y}{x} = \pm \frac{b}{a},$$

or

$$y = \pm \frac{b}{a}x. \tag{17}$$

These lines are called the *asymptotes* of the hyperbola; they are of great importance both in tracing the curve and in studying its properties. (In Chapter IX a precise definition of the term "asymptote" will be laid down, and a more detailed study of these lines will then be made.)

From the fact that the slopes of the asymptotes of the standard hyperbola are $\pm \frac{b}{a}$ we derive the following result, which gives a convenient method for drawing the asymptotes of any hyperbola whose axes are given:

The asymptotes of a hyperbola are the diagonal lines of the rectangle whose center is the center of the curve and whose sides are parallel and equal to the axes of the curve.

Before tracing a hyperbola the asymptotes should always be drawn, as they are almost indispensable in obtaining an accurate sketch.

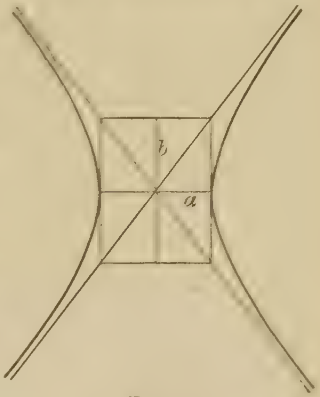


FIG. 67

79. Other standard forms. *The equation of a hyperbola with center at (h, k) is, if the transverse axis is parallel to Ox,*

$$(18) \qquad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;$$

if the transverse axis is parallel to Oy,

$$(19) \qquad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

The proof is left to the student.

To trace a hyperbola whose equation is in a standard form, we plot the vertices and the ends of the latera recta, and draw the asymptotes. These data suffice for a fairly satisfactory sketch; if greater accuracy is required, more points must be plotted.

80. General equation. The equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \tag{20}$$

where A and C have *opposite* signs, can evidently be reduced, by completing the squares in x and y, to one of the forms (18), (19) of § 79. The only exceptional case is the one in which, when the left member has been expressed as the difference of two squares, the right member reduces to 0, i.e. the case when the equation takes the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0.$$

This equation can evidently be factored in the form

$$\left(\frac{x-h}{a} + \frac{y-k}{b}\right)\left(\frac{x-h}{a} - \frac{y-k}{b}\right) = 0,$$

and therefore represents two straight lines, intersecting in the point (h, k). The way in which two real intersecting straight lines may satisfy the definition of the hyperbola will be explained in § 90. If then the pair of intersecting lines be considered as a limiting case of the hyperbola, we have the following

THEOREM: *Every equation of the second degree in which the xy-term is missing and the coefficients of x^2 and y^2 have unlike signs represents a hyperbola with its axes parallel to the coördinate axes.*

Example: Reduce the equation

$$x^2 - 2y^2 + 4x + 4y + 4 = 0$$

to a standard form, and trace the curve.

Completing the squares in x and y, we get

$$x^2 + 4x + 4 - 2(y^2 - 2y + 1) = -4 + 4 - 2,$$
$$(x+2)^2 - 2(y-1)^2 = -2,$$

or, dividing by -2,

$$\frac{(y-1)^2}{1} - \frac{(x+2)^2}{2} = 1.$$

This is a hyperbola with center at $(-2, 1)$ and transverse axis parallel to Oy; the semi-axes are $a = 1$, $b = \sqrt{2}$. The vertices are at the distance 1, the foci at the distance $\sqrt{a^2 + b^2} = \sqrt{3}$, above and below the center. The latus rectum is $\frac{2b^2}{a} = 4$, so that the ends of the latera recta are at the distance 2 to right and left of the foci. The asymptotes are the diagonals of the rectangle of sides $2a$, $2b$.

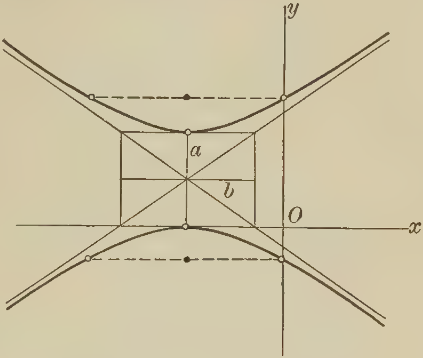


FIG. 68

The curve is shown in Fig. 68.

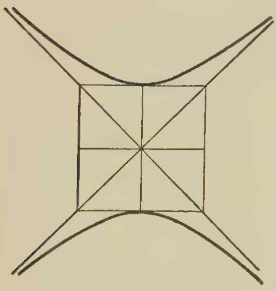


FIG. 69

81. Equilateral, or rectangular, hyperbola. The hyperbola for which a and b are equal is called, on account of the equality of the semi-axes, the *equilateral* hyperbola. It is obvious that the asymptotes of an equilateral hyperbola are at right angles, for which reason it is also called the *rectangular* hyperbola.

EXERCISES

In each of the following cases, locate the center, vertices, foci, and ends of the latera recta, draw the asymptotes, and trace the curve.

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

2. $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1.$

3. $\frac{x^2}{6} - \frac{y^2}{9} = 1.$

4. $\frac{y^2}{4} - \frac{x^2}{5} = 1.$

5. $\dfrac{(x-2)^2}{\frac{9}{4}} - \dfrac{(y+1)^2}{1} = 1.$ **6.** $\dfrac{y^2}{36} - \dfrac{(x-4)^2}{64} = 1.$

7. $\dfrac{(y-0.3)^2}{1.2} - \dfrac{(x+0.4)^2}{0.8} = 1.$ **8.** $\dfrac{(y+\frac{2}{3})^2}{6} - \dfrac{(x-\frac{7}{6})^2}{\frac{1}{4}} = 1.$

9. In Exs. 1 – 4, determine the eccentricity and write the equations of the directrices and of the asymptotes.

10. In Exs. 5–8, write the equations of the asymptotes.

11. Prove that a hyperbola cannot intersect its asymptotes.

12. By the method of § 78, derive the equations of the asymptotes of the hyperbola $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1.$

Reduce each of the following equations to a standard form, plot the center, vertices, foci, and ends of the latera recta, draw the asymptotes, and trace the curve.

13. $x^2 - y^2 = 2x.$

14. $5x^2 - 4y^2 + 20x - 24y - 36 = 0.$

15. $12x^2 - 4y^2 - 12x + 15 = 0.$

16. $9x^2 - 16y^2 - 36x + 96y + 72 = 0.$

17. $4x^2 - y^2 + 4y = 0.$

18. $4x^2 - 4y^2 - 4x + 8y - 5 = 0.$

19. $3x^2 - 2y^2 + 2y = 0.$

20. $x^2 - 5y^2 - 4x + 10y + 2 = 0.$

21. $y^2 = x^2 - 3x + 4.$

22. $x^2 - y^2 + 3x + 3y = 0.$

23. Prove that the asymptotes of an equilateral hyperbola are at right angles.

24. Show that if $C = -A$ equation (**20**) represents a rectangular hyperbola. Which of the curves in Exs. 13–22 are rectangular hyperbolas?

25. For what values of e is $b > a$? $b < a$? What is the eccentricity of the equilateral hyperbola?

[illegible] of [illegible]
indicates [illegible]

V. Coördinate Transformations

82. Introduction. We have had many illustrations of the fact that a problem may be greatly simplified by taking the axes in a convenient position. Thus to prove that the diagonals of a rectangle are equal the axes should be taken parallel to the sides of the rectangle; the properties of the circle are most easily studied if the center is taken as the origin; etc. It may happen, however, that the position of the axes is predetermined in some way by the nature of the problem, for instance by having given the equation of a curve; it is then desirable that we have methods for shifting the axes to some more suitable position. Such methods must be developed before proceeding further with the study of the conic sections.

83. Translation of axes. We will consider first the *translation* of axes, in which the axes are moved parallel to their original positions. Let Ox, Oy be the original axes, $O'x'$, $O'y'$ the new, and suppose the new origin O' to be the point (h, k) referred to the old axes. If we denote by x, y the coördinates of any point P in the original system, by (x', y') the coördinates of the same point in the new system, then it is obvious from the figure that the two sets of coördinates are connected by the formulas

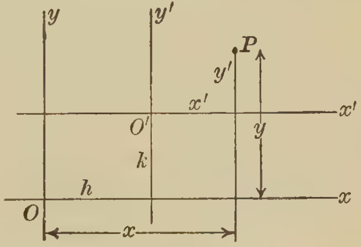


Fig. 70

$$(21)\qquad \begin{cases} x = x' + h, \\ y = y' + k. \end{cases}$$

An important problem is, being given the equation of a curve referred to any set of axes, to derive its equation referred to parallel axes through a point (h, k) as new origin. To do this, we have merely to replace x and y in the equation of the curve by their values as given by (**21**). It should be noted that this does not change the position of the curve in the plane — it merely moves the axes.

Example: From the fact that the equation of a circle with center at O is

$$x^2 + y^2 = a^2, \tag{1}$$

prove that the equation of a circle with center at (h, k) is

$$(x - h)^2 + (y - k)^2 = a^2. \tag{2}$$

If the point (h, k) be taken as the origin, O', for axes parallel to the original ones, the equation of the circle with center at (h, k) is shown by (1) to be, with reference to the new axes,

$$x'^2 + y'^2 = a^2.$$

But, by (**21**),

$$x' = x - h,\ y' = y - k;$$

therefore the equation with reference to the first axes is

$$(x - h)^2 + (y - k)^2 = a^2.$$

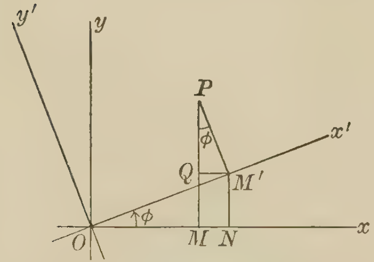


FIG. 71

84. Rotation of axes. Let Ox, Oy and Ox', Oy' be two pairs of cartesian axes with the same origin, and denote by ϕ the angle through which the first pair must be rotated to come to coincidence with the second, the angle ϕ being considered positive when measured counterclockwise. Let P have the coördinates x, y in the first system and x', y' in the second, so that, in Fig. 71,

$$OM = x,\ MP = y,\ OM' = x',\ M'P = y'.$$

Now

$$OM = ON - MN = ON - QM';$$

likewise

$$MP = MQ + QP = NM' + QP.$$

We thus have the formulas

(22) $$\begin{cases} x = x' \cos\phi - y' \sin\phi, \\ y = x' \sin\phi + y' \cos\phi. \end{cases}$$

It should be noted that the formulas of this and the preceding article taken together enable us to change the axes from any position Ox, Oy to any other position $O'x'$, $O'y'$ (Fig. 72). For we can first translate the axes to a parallel position with O' as new origin, and can then rotate them through the angle necessary to bring them to the desired position.

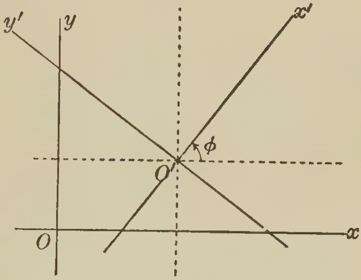


FIG. 72

Both in translation and in rotation of axes the original coördinates x, y are replaced by expressions of the *first degree* in x', y': from this we draw the important conclusion that *the degree of an equation is unchanged by translation or rotation of axes.*

Example: Rotate the axes through an angle ϕ such that the term in y will disappear from the equation

(1) $$4x + 3y = 5.$$

Substituting in (1) the values of x and y as given by (22), we get

$$4(x' \cos\phi - y' \sin\phi) + 3(x' \sin\phi + y' \cos\phi) = 5,$$

or

(2) $$(4 \cos\phi + 3 \sin\phi)x' + (-4 \sin\phi + 3 \cos\phi)y' = 5.$$

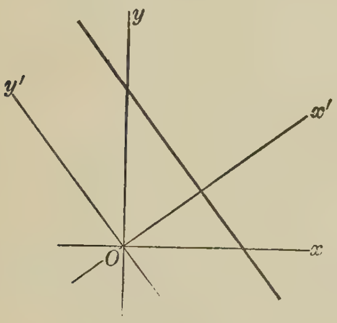


FIG. 73

To make y' drop out, we equate its coefficient to 0:

$$-4 \sin\phi + 3 \cos\phi = 0,$$

whence

$$\tan\phi = \tfrac{3}{4},$$

and

$$\sin\phi = \tfrac{3}{5},\ \cos\phi = \tfrac{4}{5}.$$

These values substituted in (2) give

$$(\tfrac{16}{5} + \tfrac{9}{5})\,x' = 5,$$

or

$$x' = 1.$$

The geometric interpretation is obvious: we have merely rotated the axes through such an angle that the new y-axis is parallel to the given line.

EXERCISES

1. Find the coördinates of the points (2, 3), (5, − 2), (0, 0) referred to parallel axes* through (− 2, 5).

2. Find the coördinates of the points (3, 4), (− 1, − 2), (x, y) referred to parallel axes through (3, 4).

3. Find the equation of the line $3x - 2y = 6$ with reference to parallel axes through (2, 3).

4. Find the equation of the line $y = mx + b$ referred to parallel axes through $(0, b)$.

5. Find the equation of the circle $x^2 + y^2 - 2ax = 0$ referred to parallel axes through $(2a, 0)$. *Ans.* $x^2 + y^2 + 2ax = 0$.

6. By translation of axes, remove the first degree terms from the equation $x^2 + y^2 - 2x + 8y + 1 = 0$.

7. Refer the line $\frac{x}{a} + \frac{y}{b} = 1$ to parallel axes through (a, b).

8. Derive (4), § 72, by translation of axes.

9. Derive (10), § 75.

10. Derive (18), § 79.

11. Find the coördinates of the point (1, 3) after the axes have been rotated through 45°. *Ans.* $(2\sqrt{2}, \sqrt{2})$.

12. Find the coördinates of the point (2, 4) after the axes have been rotated through − 30°. *Ans.* $(\sqrt{3} - 2,\ 1 + 2\sqrt{3})$.

---

* That is, axes parallel to the original ones.

13. Find the equation of the line $x + y = 3$ after rotating the axes through 45°.

14. Find the equation of the line $x \cos \beta + y \sin \beta = p$ after rotating the axes through an angle β.

15. What effect is produced upon the equation $x^2 + y^2 = a^2$ by rotating the axes through any angle ϕ?

16. Find the equation of the curve $2\,xy = a^2$ referred to axes making an angle of 45° with the original ones. *Ans.* $x^2 - y^2 = a^2$.

17. By rotating the axes remove the term in x from the equation $x + 2\,y + 3 = 0$.

18. Show that the constant term cannot be removed from the equation of a straight line by rotation of axes.

19. By rotating the axes remove the term in y from the equation $x^2 + y^2 - 2\,x - 2\,y = 0$.

20. Show in detail why the degree of an equation is unchanged by translation or rotation of axes.

VI. THE GENERAL EQUATION OF THE SECOND DEGREE

85. Introduction. In the previous sections of this chapter we have learned that every equation of the second degree in which the xy-term is missing represents a conic section with its axes parallel to the coördinate axes, and have developed methods for tracing the locus of such an equation. It remains to consider the case in which the xy-term is present.

The most general equation of the second degree has the form

$$(23) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

When the xy-term is present the methods so far used in the present chapter cannot be employed directly to trace the curve, since they depend upon reducing the equation to a standard form. When either A or C is 0 the equation can be solved rationally for one of the variables in terms of the other, so that the curve can be traced fairly well by direct point-plotting, as in Chapter II; but in the general case when all three second degree terms are present the value of either variable in terms of the other involves radicals, so that this method is very laborious. Other methods, of more general usefulness, will be developed in the next few articles.

86. Removal of the product term. The result of § 68 shows that in some cases an equation of the form

$$(1) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents a conic section, and suggests that probably it always does so. That this is the fact will be established if we can show that by a suitable coördinate transformation the term in xy can always be removed from equation (1). The transformation to be used is suggested by the following line of thought: If a conic lies with its axes inclined to the coördinate axes, for instance as in Fig. 74, we can always rotate the axes through an angle ϕ to the position Ox', Oy' parallel to the axes of the curve, and we know that when this has been done the equation of the conic must be free of the product term.

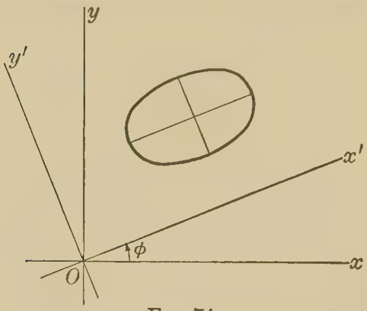


FIG. 74

It was shown in § 84 that the equation of a curve referred to axes Ox', Oy' making an angle ϕ with the original axes is found by substituting for x and y in the original equation of the curve the values

$$(2) \qquad \begin{cases} x = x' \cos\phi - y' \sin\phi, \\ y = x' \sin\phi + y' \cos\phi. \end{cases}$$

This substitution applied to equation (1) gives

$$\begin{aligned} &A(x'\cos\phi - y'\sin\phi)^2 \\ &\quad + B(x'\cos\phi - y'\sin\phi)(x'\sin\phi + y'\cos\phi) \\ &\quad + C(x'\sin\phi + y'\cos\phi)^2 + D(x'\cos\phi - y'\sin\phi) \\ &\qquad\qquad + E(x'\sin\phi + y'\cos\phi) + F = 0. \end{aligned}$$

This equation will not contain the term in $x'y'$ if we equate the coefficient of that term to 0 and determine ϕ from the resulting formula. Upon expanding and collecting terms, it appears that this gives

$$2A \sin\phi \cos\phi - 2C \sin\phi \cos\phi - B \cos^2\phi + B \sin^2\phi = 0,$$

which may be written

$$(3) \qquad (A - C)\, 2 \sin\phi \cos\phi = B (\cos^2\phi - \sin^2\phi).$$

By means of the trigonometric formulas

$$2 \sin\phi \cos\phi = \sin 2\phi, \quad \cos^2\phi - \sin^2\phi = \cos 2\phi,$$

equation (3) becomes

$$(A - C) \sin 2\phi = B \cos 2\phi,$$

or

$$\textbf{(24)} \qquad \boldsymbol{\tan 2\phi = \frac{B}{A - C}}.$$

Now for every possible value of $\tan 2\phi$, from $-\infty$ to $+\infty$, there is a value of 2ϕ between 0 and π, hence a value of ϕ between 0 and $\frac{\pi}{2}$, so that it is always possible to determine a positive acute angle ϕ satisfying* (**24**). If the curve (1) be referred to axes making this angle with the original ones, the resulting equation will be free of the product term, and its locus must be a conic. Hence we have the

THEOREM: *Every equation of the second degree represents a conic section, whose axes are inclined to the coördinate axes at an angle ϕ given by formula* (**24**).

Thus to trace the locus of any equation of the form (1) we need only to remove the xy-term by rotating the axes through the angle ϕ given by (**24**), and then to reduce the resulting equation to a standard form and proceed by the methods developed in §§ 73, 76, 80. Unfortunately, however, the values of $\sin\phi$ and $\cos\phi$ required for the formulas

---

* There are of course infinitely many other values of ϕ, differing from this one by multiples of $\frac{\pi}{2}$, but for definiteness we will agree to choose always the positive acute angle.

(**22**) are in general irrational, and the resulting algebraic complications render the process of reduction very tedious, except in especially favorable cases.

Example: Remove the xy-term from the equation

$$(4)\qquad 9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0,$$

and trace the curve.

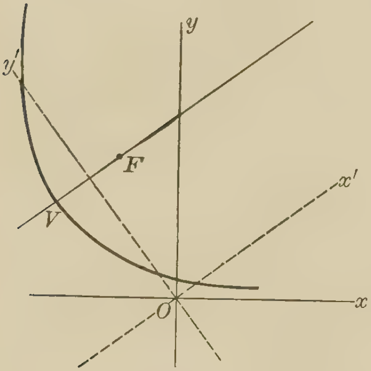


FIG. 75

Here we have

$$\tan 2\phi = \frac{-24}{9-16} = \frac{24}{7},$$

whence

$$\cos 2\phi = \tfrac{7}{25},$$

and by the formulas

$$\sin\phi = \sqrt{\frac{1-\cos 2\phi}{2}},$$

$$\cos\phi = \sqrt{\frac{1+\cos 2\phi}{2}},$$

we find

$$\sin\phi = \tfrac{3}{5},\quad \cos\phi = \tfrac{4}{5}.$$

Thus by formulas (**22**) of § 84,

$$x = \tfrac{1}{5}(4x' - 3y'),\ y = \tfrac{1}{5}(3x' + 4y').$$

In terms of the new coördinates equation (4) becomes

$$25y'^2 - 75x' - 70y' + 19 = 0,$$

or, in the standard form,

$$(y' - \tfrac{7}{5})^2 = 3(x' + \tfrac{2}{5}).$$

This is a parabola with axis parallel to Ox', opening in the positive direction, and with its vertex at the point $(-\tfrac{2}{5}, \tfrac{7}{5})$ referred to the new axes. The x'-axis has a slope $\tan\phi = \tfrac{3}{4}$; after drawing the new axes the curve is easily traced as in the figure.

87. Test for the species of a conic. It is sometimes desirable to have a test by means of which we can determine the species of a conic (whether ellipse, parabola, or hyperbola) without tracing the curve. Such a test is furnished by the following

THEOREM: *The equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents

$$\text{an ellipse if } B^2 - 4AC < 0;$$

$$\text{a parabola if } B^2 - 4AC = 0;$$

$$\text{a hyperbola if } B^2 - 4AC > 0.$$

The proof of this theorem will be deferred to § 126.

It should be noted that the condition $B^2 - 4AC = 0$ means that the quantity $Ax^2 + Bxy + Cy^2$ is a perfect square. Hence an equation of the second degree represents a parabola *if and only if the terms of the second degree form a perfect square.*

88. The rectangular hyperbola with asymptotes parallel to the coördinate axes. The equation in which both square terms are missing, viz.

$$(1) \qquad Bxy + Dx + Ey + F = 0,$$

always represents a hyperbola, since $B^2 - 4AC$ is evidently positive. Solving the equation for y,

$$y = \frac{-Dx - F}{Bx + E},$$

we see that y is infinite when $Bx + E = 0$, i.e. when $x = -\frac{E}{B}\cdot$ Thus as y increases the curve approaches the line $Bx + E = 0$ more and more closely without ever reaching it; hence this line, which is evidently parallel to Oy, is an asymptote of the hyperbola. Similarly by solving for x we find that there is an asymptote parallel to Ox. Thus we have the

THEOREM: *Every equation of the second degree in which the terms in x^2 and y^2 are both missing represents a rectangular hyperbola with its asymptotes parallel to the coördinate axes.*

When the asymptotes have been drawn, such a curve is quite easily plotted by points.

Example: Trace the hyperbola

$$xy + 2x - 3y = 12.$$

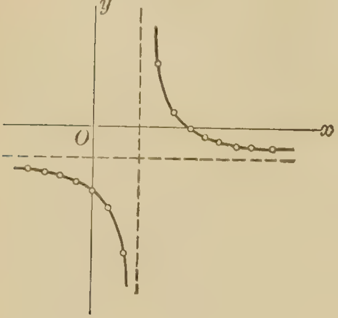


FIG. 76

Solving for y, we get

$$(2) \qquad y = \frac{12 - 2x}{x - 3},$$

whence the line $x = 3$ is an asymptote. Solving for x, we get

$$x = \frac{12 + 3y}{y + 2},$$

which shows that the line $y + 2 = 0$ is the other asymptote. Substituting values of x in (2), we compute y as follows:

x	0	1	2	3	4	5	6	7	8	9	10	-1	-2	-3	-4
y	-4	-5	-8	∞	4	1	0	$-\frac{1}{2}$	$-\frac{4}{5}$	-1	$-\frac{8}{7}$	$-\frac{7}{2}$	$-\frac{16}{5}$	-3	$-\frac{20}{7}$.

The curve is shown in the figure.

89. Curve tracing by composition of ordinates. Let it be required to trace a curve whose equation has the form

$$(1) \qquad y = u + v,$$

where u and v are any two expressions* in x. It may be possible, of course, to trace such a curve by processes already developed, but a method that is often preferable rests on the following argument. For any given value of x, the ordinate of the curve (1) is evidently the *sum of the ordinates* of the two curves

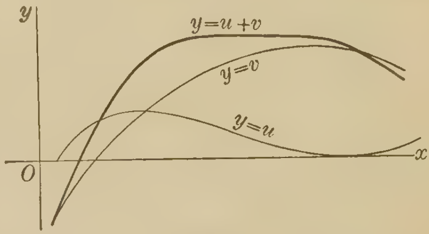


FIG. 77

$$y = u, \quad y = v.$$

---

* Of any degree whatever. The discussion is general.

If then these two curves be traced, any number of points can easily be plotted for the curve (1).

This method is particularly useful in tracing a conic whose axes are inclined to the coördinate axes. We have only to solve the equation as a quadratic in y (or in x if the term in y^2 is missing *), which exhibits y as the sum of the ordinates of two curves, one of which is a straight line, while the other is a conic with axes parallel to the coördinate axes.

Example: Trace the curve

$$x^2 - 2xy + y^2 + x - 1 = 0.$$

This curve is a parabola, since

$$B^2 - 4AC = 4 - 4 = 0.$$

Solving for y, we find

$$\begin{aligned} y &= x \pm \tfrac{1}{2}\sqrt{4x^2 - 4x^2 - 4x + 4} \\ &= x \pm \sqrt{1 - x}. \end{aligned}$$

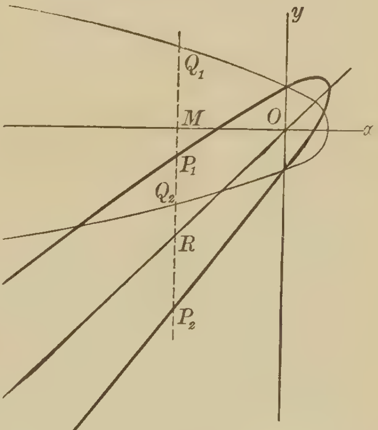


FIG. 78

The ordinates of this curve are found by adding the ordinates of the straight line

(2) $y = x$

and the curve

$$y = \pm\sqrt{1 - x},$$

that is,

(3) $y^2 = 1 - x.$

Equation (3) when reduced to a standard form is

(4) $y^2 = -(x - 1);$

this is a parabola with vertex at (1, 0), opening to the left. We now trace the curves (2) and (4) and locate points on the required curve by adding (algebraically) the ordinates of these two curves. For a given value of x there are two points on the required curve, with ordinates

---

* The case in which both square terms are missing is taken care of by § 88.

$$y = x + \sqrt{1 - x},$$
$$y = x - \sqrt{1 - x}.$$

For instance, when $x = OM$, the two ordinates are

$$MP_1 = MR + MQ_1,$$
$$MP_2 = MR + MQ_2.$$

Due regard must of course be paid to the sign of each segment involved.

EXERCISES

Trace the following curves by the method outlined in § 85.

1. $xy = 2$.

2. $y = x^2 + xy$.

3. $y^2 + 2xy = 1$.

4. $2x^2 - 3xy + 5x - y = 6$.

5. Apply the test of § 87 to the curves in Exs. 1–4.

6. Prove that if the xy-term is present and either or both square terms are missing the equation of the second degree always represents a hyperbola.

Determine the kind of conic represented by each of the following equations; remove the xy-term by rotation of axes, reduce the resulting equation to a standard form, and trace the curve.

7. $2xy + 1 = 0$.

8. $5x^2 - 6xy + 5y^2 = 8$.

9. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$.
Ans. $(x - 1)^2 = 6(y + \frac{1}{6})$.

10. $11x^2 - 24xy + 4y^2 + 6x + 8y - 10 = 0$.
Ans. $4y^2 - (x - 1)^2 = 1$.

11. $(x - y)^2 = 2(x + y)$.

12. $2x^2 - 3xy + 2y^2 = 7$.

13. $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y + 4 = 0$.
Ans. $x^2 + 4y + 1 = 0$.

14. $5x^2 + 8xy + 5y^2 + \sqrt{2}x - \sqrt{2}y = 0$.
Ans. $9x^2 + (y - 1)^2 = 1$.

15. $4xy - 3y^2 = 8$.

16. $19x^2 + 6xy + 11y^2 = 20$.

17. Prove that the equation $x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = \pm a^{\frac{1}{2}}$ represents a parabola, and trace the curve.

18. Prove that if A and C have opposite signs the equation of the second degree always represents a hyperbola. Is the converse true?

19. What is the condition that the general conic (**23**), § 85, be symmetric with respect (*a*) to Ox? (*b*) to Oy? (*c*) to O?

20. Prove that if the first degree terms are missing the equation of the second degree represents a conic with center at O. Is there any exception?

21. Prove that if the terms of the first degree are missing the equation of the second degree cannot represent a parabola. (Can it represent the limiting form of the parabola — viz. two parallel lines?)

22. By means of formula (**24**) and the theorem of § 87, deduce the theorems of §§ 73, 76, 80.

23. Prove that a conic is determined by five points.

Trace each of the following rectangular hyperbolas by drawing the asymptotes and plotting a number of points.

24. $xy + 6 = 0$.

25. $xy + 4y - 3 = 0$.

26. $3xy + 2x + 5y + 1 = 0$.

27. $2xy - x = 5$.

28. $xy - 6x - 8y + 20 = 0$.

29. $4xy - x - y = 0$.

Trace the following curves by composition of ordinates.

30. $2x^2 - 2xy + y^2 = 1$.

31. $5x^2 - 4xy + y^2 = 4$.

32. $x^2 + 2xy + y^2 - 4x - 2y = 0$.

33. $8x^2 + 12xy + 9y^2 + 4x + 6y = 0$.

34. $2xy - y^2 - 6x + 13 = 0$.

35. $x^2 - 4xy + 5y^2 - 20x + 42y + 100 = 0$.

36. $4x^2 - 4xy = 1$.

37. $3x^2 - 2xy + y^2 - 5x + 4y + 3 = 0$.

38. $3x^2 + 6xy - 9y^2 + 2x - 18y = 9$.

39. $x^2 - 6xy + 10y^2 + 8x - 23y + 10 = 0$.

40. Trace the curve of Ex. 17 by compounding ordinates.

90. Limiting forms. The conic section has been defined geometrically in § 66; analytically it is the locus of an equation of the second degree. We have also seen that an equation of the second degree may represent a circle, a pair of straight lines, or an entirely imaginary locus; it remains to show how these limiting forms may arise as a consequence of the geometric definition.

(*a*) *The circle.* Keeping the major axis of an ellipse fixed, let the eccentricity approach 0. Then b approaches a, since

$$b^2 = a^2(1 - e^2),$$

and the ellipse approaches the circular form. At the same time the foci approach the center, while the directrices recede farther and farther. Thus we may say that the circle is the conic obtained when the focus is kept in the finite plane and the directrix is supposed to have receded indefinitely.

(*b*) *The pair of lines.* Let the focus F lie on the directrix λ, as in Fig. 79. Since

$$e = \frac{FP}{LP},$$

it appears that

$$\sin\alpha = \frac{1}{e}, \tag{1}$$

where

$$\alpha = \angle LFP.$$

Thus the angle α is constant for all positions of P, and the locus therefore consists of all the straight lines that can be drawn through F making with the directrix an angle α whose sine is $\frac{1}{e}$.

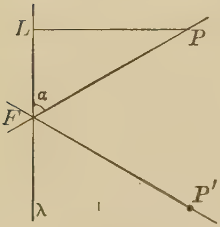


Fig. 79

First, if $e > 1$, there are evidently two such lines (the lines PF and $P'F$ in Fig. 79), the two values of α being supplementary. Thus the pair of real intersecting straight lines is a special case of the hyperbola.

Next, keeping the points P, P' fixed, let the directrix and focus recede to infinity. Then α approaches 90°, e

approaches 1, and the two lines become parallel: two parallel lines are a special case of the parabola. It should be noted that two *coincident* lines are obtained with a focus and directrix in the finite part of the plane, by merely taking $e = 1$.

Finally, if $e < 1$, there is no value of α satisfying equation (1), and the locus consists of two imaginary lines, intersecting in the real point F: a limiting form of the ellipse.

(c) *The imaginary ellipse.* The entirely imaginary locus, mentioned in § 76, can only be obtained by taking an imaginary focus and directrix. For instance, if in the work of § 74 the semi-axis a be replaced by ki (where $i = \sqrt{-1}$), so that the focus is the point $(kei, 0)$ and the directrix the line $x = \frac{ki}{e}$, the equation of the ellipse turns out to be

$$\frac{x^2}{k^2} + \frac{y^2}{b^2} + 1 = 0,$$

where

$$b^2 = k^2 (1 - e^2).$$

91. The pair of straight lines. We will now derive the analytic condition that the general equation of the second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

can be factored, and hence that the conic breaks up into two straight lines.

By § 21, if the locus of equation (1) consists of the two straight lines

$$y = mx + k, \quad y = m'x + k',$$

that equation can be written in the factored form

$$C (y - mx - k) (y - m'x - k') = 0, \tag{2}$$

and conversely. Now if we substitute the value $mx + k$ for y in (2) the equation is satisfied identically; the same result must occur if we make the same substitution in (1), since (1) is merely the expanded form of (2). Replacing y by $mx + k$ in (1), we get

$$Ax^2 + Bmx^2 + Bkx + Cm^2x^2 + 2\,Cmkx + Ck^2 + Dx + Emx + Ek + F = 0,$$

or after collecting terms,

$$(Cm^2 + Bm + A)\,x^2 + (Bk + 2\,Cmk + D + Em)\,x + Ck^2 + Ek + F = 0.$$

This is an identity if

$$Cm^2 + Bm + A = 0, \tag{3}$$

$$Bk + 2\,Cmk + D + Em = 0, \tag{4}$$

$$Ck^2 + Ek + F = 0. \tag{5}$$

Equations (3) and (5) give respectively

$$m = -\frac{B}{2\,C} \pm \frac{1}{2\,C}\sqrt{B^2 - 4\,AC},$$

$$k = -\frac{E}{2\,C} \pm \frac{1}{2\,C}\sqrt{E^2 - 4\,CF};$$

substituting these values in (4) we get

$$\frac{1}{2\,C}\sqrt{B^2 - 4\,AC}\,\sqrt{E^2 - 4\,CF} + D - \frac{BE}{2\,C} = 0,$$

or after isolating the radical, clearing of fractions, and squaring,

$$(B^2 - 4\,AC)\,(E^2 - 4\,CF) = (2\,CD - BE)^2.$$

If we remove parentheses and divide by $4\,C$ this reduces to

$$F\,(B^2 - 4\,AC) + AE^2 + CD^2 - BDE = 0, \tag{25}$$

which is therefore *the condition that the equation of the second degree shall represent two straight lines.*

In the above discussion it is tacitly assumed that $C \neq 0$. The process is easily modified to take care of the case in which $C = 0$, $A \neq 0$, and the same result is obtained. If $A = C = 0$, (1) has the form

$$Bxy + Dx + Ey + F = 0. \tag{6}$$

This equation, if factorable, can evidently be written in the form*

$$(7) \qquad B(x - h)(y - k) = 0.$$

Since (7) is true identically when $x = h$ (we might equally well, of course, use $y = k$), the same must be true of (6). The equation

$$Bhy + Dh + Ey + F = 0$$

is an identity if and only if

$$Bh + E = 0,$$
$$Dh + F = 0.$$

Eliminating h, we get

$$BF - DE = 0,$$

which is the form to which (**25**) reduces when $A = C = 0$. Thus the above result is true in all cases.

Being given a factorable equation of the second degree, it is usually desirable actually to find the factors. This may be done by solving the equation as a quadratic in y (or in x if the term in y^2 is missing). This method fails when both square terms are missing, but in that case the two factors can always be found easily by inspection (cf. equation (7)).

Example: Show that the equation

$$2x^2 + 7xy - 4y^2 + 7x + y + 3 = 0$$

represents two straight lines, and find the lines.

Formula (**25**) gives

$$3(49 + 32) + 2 - 196 - 49 = 0,$$

hence the equation represents two lines. To find the equations of the lines, we will solve for y:

$$y = \frac{-7x - 1 \pm \sqrt{(7x + 1)^2 + 16(2x^2 + 7x + 3)}}{-8}$$
$$= \frac{-7x - 1 \pm \sqrt{81x^2 + 126x + 49}}{-8}$$
$$= \frac{-7x - 1 \pm (9x + 7)}{-8},$$

---

* Since B is necessarily different from 0.

which gives

$$-8y = -7x - 1 + (9x + 7)$$

and

$$-8y = -7x - 1 - (9x + 7),$$

or finally

$$x + 4y + 3 = 0,$$
$$2x - y + 1 = 0.$$

The result may be checked by showing that these last two equations when multiplied together give the original equation.

It should be noted that y must be expressible rationally in terms of x (i.e. without having x under the radical sign), so that when the equation is solved for y the quantity under the radical sign must be a perfect square; this fact gives a preliminary check on the work.

92. Conic through five points. The equation

$$(1) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

contains five essential constants, viz. the ratios of any five of the quantities A, B, C, D, E, F to the sixth. It follows that a conic may be made to pass through any five points. Substituting the coördinates of these points in (1) in turn, we obtain five linear equations to determine the five ratios; thus through five given points only one conic, in general, may be drawn.

To find the equation of the conic determined by five points, we might, of course, actually write out the five linear equations above mentioned and solve them for five of the constants in terms of the sixth. The following method, however, is somewhat shorter.

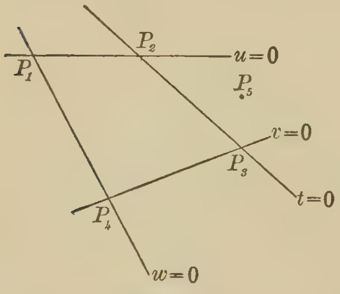


FIG. 80

Let the five given points be P_1, P_2, P_3, P_4, P_5. For the present we will leave one of the points, say P_5, out of consideration. Write the equation of the line through the points P_1, P_2 and denote its equation by

(1) $$u = 0,$$

where u of course is a certain expression of the first degree in x and y. Again, write the equation of the line through P_3, P_4; call it

(2) $$v = 0.$$

Similarly write the equations of another pair of lines through the four points, say

(3) $$w = 0$$

through P_1, P_4, and

(4) $$t = 0$$

through P_2, P_3.

Now let us multiply together equations (1) and (2), forming the equation

(5) $$uv = 0;$$

similarly form the equation

(6) $$wt = 0.$$

By § 21, the locus of equation (5) is the two straight lines (1) and (2) taken together, while (6) represents the lines (3), (4).

Next, form the equation

(7) $$uv + kwt = 0,$$

where k is an undetermined constant. By § 57, the locus of (7) is a curve passing through the points of intersection of the curves (5) and (6), i.e. through the points P_1, P_2, P_3, P_4. But since equation (7) is evidently of the second degree it represents a conic section. If we determine k so that this curve shall pass through P_5, it will be the desired conic.

93. Parabola through four points. In the case of the parabola, one relation is given among the constants, viz.

$$(1) \qquad B^2 - 4AC = 0;$$

hence a parabola can be passed through only four points. Since this relation is of the second degree in the constants, while the four equations arising from the substitution of the coördinates are linear, it follows that through any four points two parabolas may be drawn, in general. These parabolas may of course be real and distinct, real and coincident, or imaginary.

To determine the parabolas through four points by the method of § 92, we form the equation

$$uv + kwt = 0$$

as in that article and then determine k by the condition (1).

EXERCISES

Show that the following equations represent pairs of lines, and find the equations of the lines.

1. $x^2 - y^2 + 4x - 10y = 21$. *Ans.* $x - y = 3, x + y + 7 = 0$.

2. $6x^2 - xy - y^2 + 12x - 6y = 0$.

3. $2xy + 10x - 3y = 15$.

4. $x^2 + y^2 - 2x + 10y + 26 = 0$. *Ans.* $y = -5 \pm i\,(x - 1)$.

5. $2x^2 - 4xy + 7x - 2y + 3 = 0$.

6. $y^2 - 6xy + 9x^2 = 6x - 2y$.

7. Determine k so that the equation $3x^2 - 3xy - y^2 - x + k = 0$ shall represent a pair of lines.

8. Determine k so that the equation $2x^2 + 2y^2 - 4y + k = 0$ shall represent two straight lines.

Find the equations of the following conics; determine the species, and check the results.

9. Through (0, 0), (1, 0), (0, 2), (3, 1), (– 1, 2).

Ans. $x^2 + xy + 9y^2 - x - 18y = 0$.

10. Through (2, 1), (– 2, 1), (1, 2), (– 1, – 2), (– 2, – 1).

11. Through (2, 4), (– 3, – 1), (0, 5), (3, 2), (3, 3).
Ans. $y^2 + 2x - 5y = 0$.

12. Through (5, 3), (– 1, 6), (3, 0), (1, 2), (8, 4).
Ans. $158x^2 - 381xy - 314y^2 + 218x + 1859y - 2076 = 0$.

13. Parabola through (0, 0), (9, – 3), (1, 1), (4, 2).
Ans. $y^2 = x$; $x^2 - 7x + 6y = 0$.

14. Parabola through (2, 1), (6, 9), (– 4, 4), (– 12, 36).

15. Parabola through (1, 1), (7, – 2), (3, 2), (7, 3).

16. Prove analytically that a pair of lines is determined by four points, and that through four points three pairs of lines can be drawn.

CHAPTER VIII

THE PARABOLA

94. Introduction. In the previous chapter we have learned how to write the equation of a conic from the definition, and have also developed methods for tracing the curve when its equation is given. It remains to investigate the properties of the conics by analytic methods; this will be done for the parabola in the present chapter.

For convenience, the geometric properties of the curve — i.e. those properties that depend upon the form of the curve and not upon its position with regard to the coördinate axes — will as a rule be deduced by use of the equation

$$y^2 = 4\,ax.$$

This of course does not impair the generality of the results.

We begin with a list of exercises that can be solved by means of the theory already developed.

EXERCISES

In Exs. 1–5, in addition to the given data, how many points are required to determine the parabola? Give reasons.

1. Vertex and directrix given.

2. Vertex and axis given.

3. The direction of the axis given.

4. The vertex given.

5. The axis given.

Find the equations of the following parabolas.

6. With vertex at O, axis coinciding with Ox, and passing (*a*) through $(-1, -6)$; (*b*) through $(4, -2)$.

7. With vertex at O and focus $(0, -2)$.

8. With vertex $(2, 3)$ and focus $(6, 3)$.

9. With vertex $(-1, -2)$ and focus $(-\frac{3}{2}, -2)$.

10. With vertex $(3, 4)$ and directrix $x = 4$.

11. With vertex $(0, 2)$ and directrix $3x + 2 = 0$.

12. With vertex $(6, -1)$ and directrix $4y - 3 = 0$.

13. With vertex $(2, 5)$, axis parallel to Ox, and latus rectum 3.

14. With focus $(1, 6)$, axis parallel to Oy, and latus rectum 6.

15. With vertex $(-2, 1)$, axis parallel to Ox, and passing through $(2, 2)$.

16. With vertex on Oy, axis parallel to Ox, and passing through $(\frac{1}{2}, 3)$, $(2, 4)$.

17. With vertex on the line $x = 2$, axis parallel to Ox, latus rectum 6, and passing through $(8, 3)$.

18. With axis $x = 3$, and passing through $(2, -\frac{9}{2})$, $(5, -6)$.

Ans. $(x - 3)^2 = -2(y + 4)$.

19. With axis parallel to Oy, and passing through $(0, 0)$, $(1, 3)$, $(4, -2)$.

20. With axis parallel to Ox, and passing through $(2, 0)$, $(-1, 1)$, $(-2, 2)$.

Ans. $y^2 - x - 4y + 2 = 0$.

21. Find the locus of the center of a circle which passes through $(2, 1)$ and touches the line $x = 10$.

22. Find the equation of a circle which passes through the points $(0, -5)$, $(-3, -4)$ and touches the line $y = 5$.

Ans. $x^2 + y^2 = 25$; $(x + 60)^2 + (y + 180)^2 = (185)^2$.

23. Find the locus of the center of a circle which touches the circle $x^2 + y^2 = 16$ and the line $x = 6$. Trace the locus.

Ans. $y^2 = -20(x - 5)$.

24. Find the locus of the center of a circle which touches the line $x = 2$ and the circle $x^2 + y^2 = 16$. Trace the curve.

Ans. $y^2 = -12(x - 3)$, $y^2 = 4(x + 1)$.

25. Find the locus of the center of a circle which touches the circle $x^2 + y^2 = 2x + 1$ and the line $x + y + 10 = 0$.

Ans. $x^2 - 2xy + y^2 - 20x - 16y - 62 = 0$.

95. Similar and equal geometric figures. In geometry, two sets of points $P_1, P_2, P_3, \cdots, Q_1, Q_2, Q_3, \cdots$ are said to be *similar* if they can be so placed with reference to a point O that every line joining O to a point of the first set passes through the corresponding point of the second set — i.e. that the lines OP_1, OP_2, OP_3, $\cdots$ pass through $Q_1, Q_2, Q_3, \cdots$ respectively — and that further the point O divides each of the segments P_1Q_1, P_2Q_2, P_3Q_3, $\cdots$ in the same ratio — i.e. that

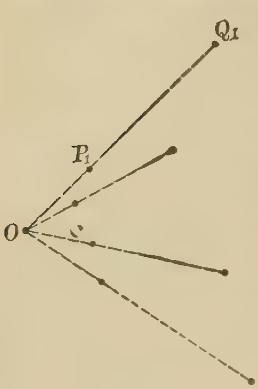


FIG. 81

$$\frac{OP_1}{OQ_1} = \frac{OP_2}{OQ_2} = \frac{OP_3}{OQ_3} = \cdots = k,$$

where k is a constant. When the points are actually placed in such a way that these conditions are satisfied they are said to be *similarly placed*, and the point O is called the *center of similitude*. Figure 81 shows two similar sets of points similarly placed with respect to the point O as center, the ratio k being equal to $\frac{1}{3}$.

Two geometric figures of any kind are said to be *similar*, or to *have the same shape*, when the sets of points composing the figures are similar. For example, all circles are similar; two triangles are similar if their angles are equal; etc.

Two similar figures are said to be *equal* [*] if they have the *same size* — i.e. if they can be made to coincide by superposition. Two circles are equal if their radii are equal; two triangles are equal if the three sides of one equal respectively the three sides of the other.

96. Similarity and equality of parabolas. By § 69, the polar equation of a parabola with focus at the pole and axis coinciding with the initial line is

---

[*] In some texts the word *congruent* is used instead.

$$r = \frac{l}{1 + \cos\theta}.$$

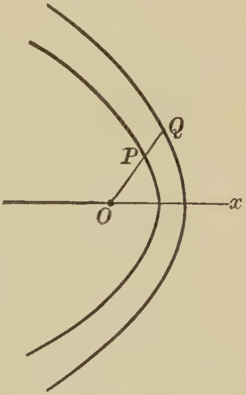


FIG. 82

Let

(1) $$r = \frac{l_1}{1 + \cos\theta}$$

and

(2) $$r = \frac{l_2}{1 + \cos\theta}$$

be the equations of any two parabolas in this position. Let P and Q be the points on the two curves corresponding to any value of θ; then by (1) and (2) the corresponding values of r are

$$OP = \frac{l_1}{1 + \cos\theta}$$

and

$$OQ = \frac{l_2}{1 + \cos\theta}.$$

Dividing, we find

$$\frac{OP}{OQ} = \frac{l_1}{l_2};$$

since $\frac{l_1}{l_2}$ is constant (i.e. independent of θ), we have proved the

THEOREM: *All parabolas are similar; if two parabolas have the same focus and axis they are similarly placed, with the common focus as center of similitude.*

Since all parabolas have the same shape, it is evident that the size is determined by a single magnitude, e.g. the distance from the vertex to the focus; in other words, two parabolas are equal if the distance from the vertex to the focus is the same for each.

97. Geometric problems involving the parabola. An important part of elementary geometry consists of the solution of so-called "problems," in which it is required to make certain geometric constructions from given data. In the solution of problems two fundamental assumptions are made, as follows:

A straight line can be drawn if two of its points are given.

A circle can be drawn if its center and radius are given.

That is, it is assumed that the straight-edge and compass can be used.

A variety of interesting geometric problems arise in connection with the parabola, in many of which we have actually to draw the curve. While any number of points on a parabola can be constructed by means of the ruler and compass if the focus and directrix are given, yet those instruments do not suffice to trace a continuous arc of the curve; it is therefore necessary to assume that some new instrument or device is available by means of which the curve may be drawn*. That is, we lay down a third assumption:

A parabola can be drawn if its focus and directrix are given.

Examples: (*a*) Given the axis, vertex, and one point of a parabola, construct the curve (i.e. find the focus and directrix).

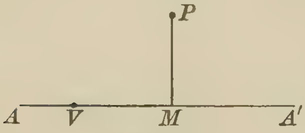


FIG. 83

Let AA' be the axis, V the vertex, and P a point on the curve. The required construction suggests itself as follows: Drop a perpendicular PM from

---

* A device for tracing an arc of a parabola is as follows: Place the base AB of a right-angled triangle ABC upon the directrix λ; take a string of length BC and fasten its ends at C and F. Then as the base of the triangle slides along the directrix a pencil keeping the string taut at P traces an arc of the curve. For,

$$CB = CP + PF,$$

or

$$CB - CP = PF;$$

that is,

$$PB = PF.$$

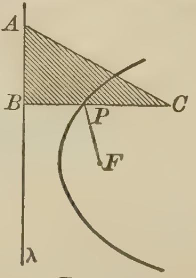


FIG. 84

P to the axis, and denote VM by x, MP by y. Then we know that

$$(1) \qquad y^2 = 4\,ax,$$

where a is the unknown distance from V to the focus. Our problem thus takes this form: If x and y are given lengths, solve equation (1) geometrically for a.

Now if we write (1) in the form

$$\frac{x}{y} = \frac{y}{4\,a},$$

it appears that the latus rectum $4\,a$ is a third proportional to x and y. To find a third proportional to two given segments is a familiar problem of elementary geometry. The remainder of the construction, and the proof, are left to the student.

It should be observed that the above discussion is couched in the language of analysis merely for convenience: the argument is entirely independent of any coördinate system. The service rendered by analysis lies further back, viz. in helping us to establish the characteristic geometric property of the parabola represented by (1) (cf. § 72).

(b) Construct a circle to pass through two given points and touch a given line*.

Let P_1, P_2 be the given points and λ the given line.

Construction: Erect the perpendicular bisector AB of the line joining the given points.

Draw the parabola having P_1 as focus and λ as directrix.

The points C_1, C_2 where AB intersects the parabola are the centers of two circles satisfying the given conditions.

The proof is left to the student.

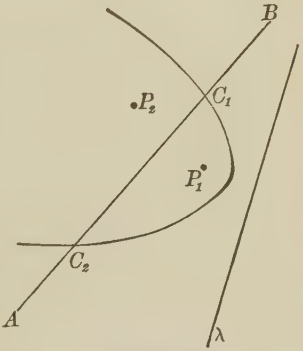


FIG. 85

---

* This problem can be solved by the ruler and compass, but the method is much less obvious than the one here used.

EXERCISES

1. Show that any two straight line segments are similar, and that they may be similarly placed in infinitely many positions — i.e. that they have an infinite number of centers of similitude.

2. Prove (*a*) geometrically, (*b*) analytically, that if two circles are tangent to each other, the point of contact is a center of similitude.

3. Prove that the point of intersection of their common tangents is a center of similitude for two circles.

4. How many quantities (angles, lengths, etc.) must be given to determine each of the following figures in shape? In both shape and size?

(*a*) A square; (*b*) a triangle; (*c*) a rectangle;
(*d*) a sphere; (*e*) a right circular cone; (*f*) a cube.

5. Derive the polar equation of a parabola with vertex at the pole and focus at $(a, 0)$, (*a*) directly, (*b*) by transforming the equation $y^2 = 4ax$. *Ans.* $r \sin\theta \tan\theta = 4a$.

6. Using the result of Ex. 5, prove that two parabolas having the same vertex and axis are similarly placed, with the common vertex as center of similitude.

7. Derive the polar equation of a parabola with the directrix as initial line and the focus at $\left(2a, \frac{\pi}{2}\right)$.

Ans. $r^2\cos^2\theta - 4ar\sin\theta + 4a^2 = 0$.

8. Prove that two parabolas having the same directrix and axis are similarly placed. Where is the center of similitude?

9. Show in detail how to construct points of a parabola if the focus and directrix are given.

10. Complete the construction and give the proof in example (*a*), § 97.

11. Given a parabola with its vertex, construct the axis.

12. Given a parabola with its focus, find the directrix.

13. Given a parabola with its directrix, find the focus.

14. Given the directrix and two points of a parabola, find the focus. How many solutions are there, in general? Discuss exceptional cases.

15. Given the focus and two points of a parabola, find the directrix. How many solutions are there?

16. Given the directrix, the tangent at the vertex, and one point, construct the focus.

17. If x, y, and b are given lengths, solve the equation $\frac{x}{a} + \frac{y}{b} = 1$ geometrically for a.

18. Given the lengths a, b, and y, solve the equation $\frac{x}{a} + \frac{y}{b} = 1$ geometrically for x.

19. Find y geometrically from the equation $y = mx + b$, if x and b are given lengths and m is the ratio of two other lengths.

20. Find y geometrically from the equation $y^2 = 4ax$, if a and x are given distances.

21. Given the axis, focus, and one point of a parabola, construct the directrix. (Use the property $r = \frac{l}{1 + \cos\theta}$.)

22. Given a parabola and its vertex, find the focus without using the property $y^2 = 4ax$.

23. Give the proof in example (*b*), § 97.

24. Solve example (*b*), § 97, by another method.

25. In example (*b*), § 97, discuss the following cases:

(*a*) P_1, P_2 on opposite sides of λ;
(*b*) P_1 lying on λ;
(*c*) λ perpendicular to P_1P_2;
(*d*) λ perpendicular to P_1P_2 at P_1;
(*e*) λ parallel to P_1P_2.

26. Draw a circle of given radius passing through a given point and touching a given line. Solve by two methods.

27. Construct a circle passing through a given point and touching two given lines. How many solutions are there? Discuss special cases.

28. Construct the locus of the center of a circle touching a given line and a given circle. Consider the cases in which the given line and circle (*a*) do not intersect, (*b*) do intersect.

29. Draw a circle touching two given lines and a given circle. Determine the number of solutions in each of the following cases:

(*a*) neither line intersecting the given circle;
(*b*) one line intersecting the circle;
(*c*) both lines intersecting the circle.

98. Tangent at a given point. The tangent to a parabola, or in fact to any conic, at a given point on the curve can be found by the condition for tangency (§ 52). It is easy, however, to devise a method based directly on the definition of tangent (§ 51), which is somewhat shorter, and much more general in that it applies to curves of any degree as readily as to the conics. We will begin by applying this method to an

Example: Find the tangent to the parabola

$$(1) \quad y^2 - 3x - y + 1 = 0$$

at the point P: (1, 2).

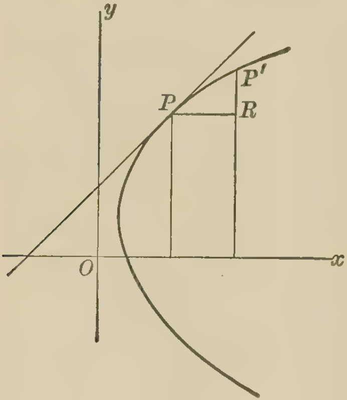


FIG. 86

Choose a point P' on the curve near the given point, and denote the distances PR, RP' (Fig. 86) by* Δx, Δy respectively, so that the coördinates of P' are $(1 + \Delta x, 2 + \Delta y)$. Since P' lies on the curve, its coördinates may be substituted for x and y in equation (1). This gives

$$(2 + \Delta y)^2 - 3(1 + \Delta x) - (2 + \Delta y) + 1 = 0,$$

which reduces to

$$3\,\Delta y + \overline{\Delta y}^2 - 3\,\Delta x = 0.$$

If this equation be divided through by Δx, it may be written in the form

$$(2) \qquad 3\frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x}\Delta y - 3 = 0.$$

The quantity $\frac{\Delta y}{\Delta x}$ occurring here is evidently the slope of the secant PP'. If now we let Δx approach 0, so that P' approaches P along the curve, Δy will also approach 0, but by § 51 *the ratio* $\frac{\Delta y}{\Delta x}$ *will approach as its limit the slope* m *of the curve at* P. Hence, when Δx approaches 0, (2) becomes

---

* Read: delta x.

$$3\,m - 3 = 0.$$

(In the second term, $\frac{\Delta y}{\Delta x}\,\Delta y$, the factor $\frac{\Delta y}{\Delta x}$ approaches m while the factor Δy approaches 0, so that the whole term approaches 0.) Thus

$$m = 1,$$

and the equation of the tangent is

$$y - 2 = x - 1.$$

The essential steps of the method may now be summarized as follows:

To find the slope m *of a given curve at a point* $P\colon (x_1, y_1)$ *on the curve, choose a neighboring point* $P'\colon (x_1 + \Delta x, y_1 + \Delta y)$ *on the curve, substitute the coördinates of* P' *in the equation of the curve, and simplify. Divide through by* Δx. *Let* Δx *approach* 0, $\frac{\Delta y}{\Delta x}$ *at the same time approaching the value* m, *and solve for* m.

As a further application of this method let us find the tangent to the parabola

$$y^2 = 4\,ax \tag{3}$$

at any point (x_1, y_1) on the curve. Substituting the coördinates of $P'\colon (x_1 + \Delta x, y_1 + \Delta y)$ in (3), we get

$$y_1^2 + 2\,y_1\,\Delta y + \overline{\Delta y}^2 = 4\,ax_1 + 4\,a\,\Delta x. \tag{4}$$

Since (x_1, y_1) lies on the curve we have

$$y_1^2 = 4\,ax_1, \tag{5}$$

whence (4) becomes

$$2\,y_1\,\Delta y + \overline{\Delta y}^2 = 4\,a\,\Delta x,$$

or

$$2\,y_1\frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x}\,\Delta y = 4\,a.$$

When Δx approaches 0, this reduces to

$$2\,y_1 m = 4\,a,$$

or

$$m = \frac{2a}{y_1}.$$

The desired equation is therefore

$$y - y_1 = \frac{2a}{y_1}(x - x_1),$$

or

$$y_1y - y_1^2 = 2ax - 2ax_1.$$

Simplifying by means of (5), we obtain the result:

The equation of the tangent to the parabola

$$y^2 = 4ax$$

at the point (x_1, y_1) *on the curve is*

$$\boldsymbol{y_1y = 2ax + 2ax_1.} \tag{1}$$

99. Normal; subtangent; subnormal. A line perpendicular to the tangent at a point of a curve is called the *normal* to the curve at that point. The equation of the normal can evidently be written down at once when that of the tangent is known.

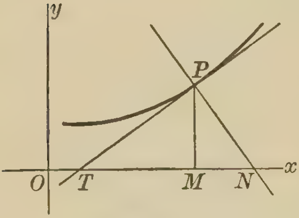


Fig. 87

If the tangent and normal at a point P of a plane curve cross the x-axis at T and N respectively, and if M is the foot of the ordinate of P, then the distances TM and MN are called the *subtangent* and the *subnormal* corresponding to the point P.

100. Subtangent and subnormal for the parabola. By § 98, the tangent to the parabola

$$y^2 = 4ax$$

at P: (x_1, y_1) is

$$y_1y = 2ax + 2ax_1.$$

The x-intercept of this line is

$$OT = -x_1,$$

so that the subtangent is

$$TM = 2x_1.$$

We have thus proved that the subtangent for the parabola $y^2 = 4ax$ is *bisected at the vertex.*

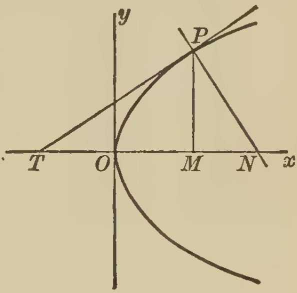


FIG. 88

The slope of the tangent at P is $\frac{2a}{y_1}$; therefore the equation of the normal at P is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

The x-intercept of this line is

$$ON = x_1 + 2a,$$

so that the subnormal is

$$MN = 2a:$$

i.e. *the subnormal is constant,* and equal to the distance from the directrix to the focus.

EXERCISES

Find the tangent and normal to each of the following curves at the points indicated.

1. $y^2 + 3x = 0$ at $(-3, 3)$.
2. $3y^2 = 2x$ at $(6, -2)$.
3. $x^2 + 4y = 0$ at $(-2, -1)$.
4. $2y^2 - 5x + 4y + 15 = 0$ at $y = -2$.
5. $y = 3x^2 - x$ at the origin.
6. $(x+1)^2 = 2(y-3)$ at $(1, 5)$.
7. $(y - \frac{1}{2})^2 = -(x + \frac{1}{2})$ at $y = 0$.
8. $x^2 - 4xy + 4y^2 + 2x = 0$ at the origin. *Ans.* $x = 0$.
9. $(3x - y)^2 + 2x + y = 5$ at $(1, 3)$. *Ans.* $2x + y = 5$.
10. $x^2 = 4ay$ at (x_1, y_1).
11. $y^2 + 4x - 6y + 2 = 0$ at (x_1, y_1).
    Ans. $y_1 y + 2(x + x_1) - 3(y + y_1) + 2 = 0$.
12. $2x^2 - 3x - 5y = 0$ at (x_1, y_1).
    Ans. $4x_1x - 3(x + x_1) - 5(y + y_1) = 0$.

13. $y = x^3 - 3x^2 + 2x - 2$ at $(1, -2)$. *Ans.* $x + y + 1 = 0$.

14. Find the slope of the curve $y = 2x^3 + 4x^2 - x + 3$ at (x, y).
Ans. $6x^2 + 8x - 1$.

15. By the method of § 98, find the tangent to the circle $x^2 + y^2 = 10$ at the point $(3, 1)$.

16. Find the tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1).

17. Prove that every normal to a circle passes through the center.

Find the angle between the following curves at each of their points of intersection.

18. $y^2 = x, \quad 2y = x$. *Ans.* $\phi = \arctan 2$; $\phi = \arctan \frac{2}{9}$.

19. $(x - 2y)^2 + 3x - 2y - 12 = 0, y = x + 2$.
Ans. $\phi = \arctan \frac{9}{19}$; $\phi = \arctan \frac{9}{35}$.

20. $x^2 + y^2 = 4, \quad y^2 = 3x$. *Ans.* $\phi = \arctan \frac{5}{3}\sqrt{3}$.

21. $y^2 + 4x - 4y = 0, \quad (3x - y)^2 + x = y$.
Ans. $\phi = 0$; $\phi = \arctan \frac{3}{7}$; $\phi = \arctan \frac{75}{67}$.

22. Prove that two equal parabolas with the same focus and axis, but opening in opposite directions, intersect at right angles.

23. In Ex. 6, find the subtangent and subnormal corresponding to the given point. *Ans.* Subtangent $= \frac{5}{2}$, subnormal $= 10$.

24. In Ex. 9, find the subtangent and subnormal corresponding to $(1, 3)$.

25. For the curve $y = 1 - 2x + 3x^2 - x^3$, find the subtangent and subnormal corresponding to the point $x = 3$.

26. State the two results of § 100 in the form of general theorems applicable to all parabolas.

27. Given a parabola and its vertex, construct the tangent and normal at any point.

28. Given the axis, vertex, and one point, construct the curve.

29. Given the axis and one point of a parabola, with the tangent at that point, construct the curve.

30. Prove that any tangent to a parabola intersects the directrix and the latus rectum (produced) in points equally distant from the focus.

101. Tangents having a given slope; tangents drawn from an external point. To find a tangent to a parabola when the slope m is given, we need only assume the equation of the required tangent in the form

$$y = mx + k,$$

and determine k by the condition for tangency precisely as in § 53. Similarly we may avail ourselves of the method used in the example of § 55 to find tangents to a parabola through an external point. In fact, it is clear that all those methods of Chapter V (The Circle) that were based on the use of the quadratic discriminant may be used without the slightest change to solve similar problems not merely for the parabola, but for any conic.

As an illustration, let us determine k so that the line

$$y = mx + k, \tag{1}$$

where m is supposed known, shall touch the parabola

$$y^2 = 4\,ax. \tag{2}$$

Substituting the value of y from (1) in (2) and collecting terms, we get

$$m^2x^2 + 2\,(mk - 2\,a)\,x + k^2 = 0.$$

The condition

$$b^2 - 4\,ac = 0$$

gives

$$4\,(m^2k^2 - 4\,amk + 4\,a^2) - 4\,m^2k^2 = 0,$$

whence

$$k = \frac{a}{m}\cdot$$

Thus *the line*

$$\boldsymbol{y = mx + \frac{a}{m}} \tag{2}$$

is tangent to the parabola

$$y^2 = 4\,ax$$

for all values of m.

102. Number of tangents through a given point. Let $P: (x_1, y_1)$ be a point outside the parabola

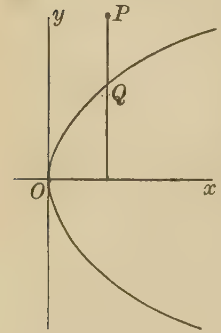


FIG. 89

$$y^2 = 4\,ax,$$

and assume first that x_1 has the same sign as a (thus if a is positive, for instance, P lies in the first or the fourth quadrant). Then the ordinate of P will intersect the parabola: denote the point of intersection by $Q: (x_1, y_2)$. Now since Q is on the parabola we have

$$y_2^2 = 4\,ax_1.$$

But since

$$y_1^2 > y_2^2,$$

it follows that

$$y_1^2 > 4\,ax_1,$$

or

(1) $$y_1^2 - 4\,ax_1 > 0.$$

If $x_1 = 0$, or if x_1 and a have opposite signs, it is obvious at once that (1) holds. Hence this inequality is true if P is any point outside the parabola.

In a similar way it can be shown that

$$y_1^2 - 4\,ax_1 < 0$$

if P is inside the parabola, while of course

$$y_1^2 - 4\,ax_1 = 0$$

if P is on the curve.

Now let

$$y = mx + \frac{a}{m}$$

be a tangent to the parabola through P. Since P is on this line, we have

(2) $$y_1 = mx_1 + \frac{a}{m},$$

or, clearing of fractions and transposing,

$$x_1m^2 - y_1m + a = 0.$$

[illegible]

$x^2 - 11xy + 3y^2 - 7 = 0$.

The roots of this quadratic in m are real and distinct, equal, or imaginary, and hence through P there are two, one, or no tangents, according as $y_1^2 - 4\,ax_1$ is positive, zero, or negative. Thus we have proved the

THEOREM: *Through a point P two tangents, one tangent, or no tangents can be drawn to a parabola according as P is outside, on, or inside the curve.*

103. Two geometric properties. The results of §§ 98, 100, 101 enable us to establish in a variety of ways the following important theorems concerning the parabola.

THEOREM I: *The tangent bisects the angle between the focal radius drawn to the point of contact and the line through that point parallel to the axis.*[1]

THEOREM II: *The foot of the perpendicular from the focus upon any tangent lies on the tangent at the vertex.*

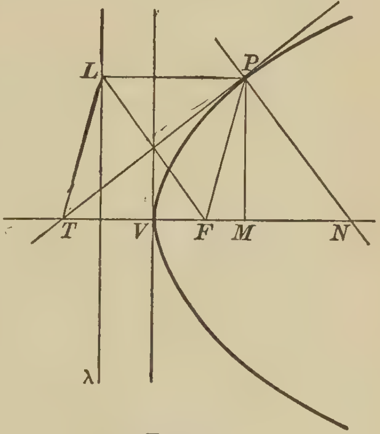


FIG. 90

That is, in Fig. 90, the line PT bisects the angle FPL, and the line FL intersects PT on the tangent at V.

The proof of these theorems is left to the student.

EXERCISES

Find the equations of the following tangents.

1. To the parabola $2\,x^2 + 3\,x = 3\,y$ parallel to the line $2\,x + 2\,y = 1$. *Ans.* $2\,x + 2\,y + 3 = 0$.

2. To the parabola $y^2 - 6\,x + 5\,y + 1 = 0$ perpendicular to the line $x + 3\,y = 5$. *Ans.* $24\,x - 8\,y + 5 = 0$.

3. To the parabola $3\,y^2 - x + 7\,y + 5 = 0$ parallel to the line $4\,x - 3\,y = 10$. *Ans.* $192\,x - 144\,y = 335$.

4. To the parabola $4x^2 - 12xy + 9y^2 + 3x - 2y = 0$ perpendicular to the line $2x + 3y + 5 = 0$. *Ans.* $3x - 2y = 0$.

5. To the parabola $(2x - y)^2 = y$ parallel to the y-axis.

6. To the parabola $x^2 = 3y$ through the point $(-2, 1)$.
Ans. $2x + y + 3 = 0, 2x + 3y + 1 = 0$.

7. To the parabola $x^2 - 5x + 4y + 6 = 0$ through the origin.
Ans. $y = (\frac{5}{4} \pm \frac{1}{2}\sqrt{6})\,x$.

8. To the parabola $2y^2 - 3x + 4y + 3 = 0$ through $(11, 4)$.

9. To the parabola $2x^2 - 4xy + 2y^2 = x$ through $(-2, -2)$.

10. To the parabola $(x - y)^2 = x$ through the point $(0, 1)$.

11. Find the equation of a parabola with vertex at O, axis coinciding with Ox, and touching the line $y = 3x + 4$. *Ans.* $y^2 = 48x$.

12. Find the equation of a parabola with vertex at $(2, 1)$, axis parallel to Ox, and touching the line $y = x + 1$.
Ans. $(y - 1)^2 = 8(x - 2)$.

13. Find the equation of a parabola with vertex at $(4, -1)$, axis parallel to Oy, and touching the line $y = 3x$.
Ans. $9x^2 - 72x + 52y + 196 = 0$.

14. Prove that only one tangent to a parabola can be drawn parallel to a given line.

15. Show that no tangent can be drawn to a parabola parallel to its axis.

16. Prove that the quantity $x_1^2 + y_1^2 - a^2$ is positive, zero, or negative according as the point (x_1, y_1) is outside, on, or inside the circle $x^2 + y^2 = a^2$.

17. Using the formula of Ex. 18, p. 83, prove that two tangents can be drawn to a circle through an external point.

18. Prove that the tangents drawn to a parabola from any point of the directrix are perpendicular.

19. State and prove the converse of the theorem of Ex. 18.

20. In Fig. 90, prove that

$$FP = FT = FN.$$

21. Prove Theorem I, § 103, geometrically.

22. Prove Theorem II, § 103, geometrically.

23. Prove Theorem II, § 103, analytically, using the formula (1) of § 98.

[illegible] one line [illegible]

[illegible] Exam

24. Prove Theorem II, § 103, by means of formula (**2**), § 101.

25. Given a parabola and its vertex, construct the tangent and normal at any point by a new method. (Ex. 27, p. 158.)

26. Given the axis and one point of a parabola, with the tangent at that point, construct the curve. (Ex. 29, p. 158.)

27. Prove in a new way that any tangent to a parabola intersects the directrix and the latus rectum (produced) in points equally distant from the focus. (Ex. 30, p. 158.)

28. Given the axis, the focus, and any tangent, construct the directrix by two methods.

29. Given the focus, and any tangent with its point of contact, construct the directrix.

30. Given the directrix and one point, with the tangent at that point, find the focus.

31. Given the tangent at the vertex, and one other tangent with its point of contact, construct the curve.

32. Given the focus and two tangents, find the vertex.

33. Given three tangents, of which one is the tangent at the vertex, construct the curve.

104. Chord of contact. If tangents are drawn to a conic through an external point P, the line through the points of contact is called the *chord of contact* corresponding to P, with reference to the given conic.

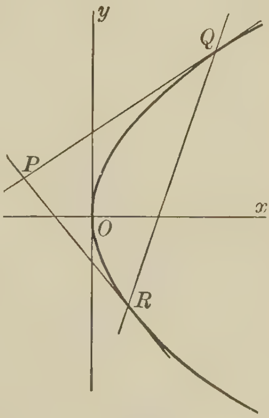


FIG. 91

To find the chord of contact corresponding to the external point P: (x_1, y_1) with respect to the parabola

$$y^2 = 4\,ax,$$

we may proceed exactly as in § 56. Let QR (Fig. 91) be the required chord, and let the coördinates of Q and R (which are of course unknown) be denoted by (x_2, y_2) and

(x_3, y_3) respectively. Then by (**1**), § 98, the equation of the tangent at Q is

$$y_2y = 2\,ax + 2\,ax_2,$$

that of the tangent at R is

$$y_3y = 2\,ax + 2\,ax_3.$$

By hypothesis, both these lines pass through P: substituting the coördinates of P for x and y in the two equations we obtain the relations

$$(1) \qquad y_2y_1 = 2\,ax_1 + 2\,ax_2,$$

$$(2) \qquad y_3y_1 = 2\,ax_1 + 2\,ax_3.$$

Consider now the straight line

$$(3) \qquad y_1y = 2\,ax + 2\,ax_1.$$

This line passes through Q, for if we substitute in (3) the coordinates (x_2, y_2) we obtain (1), which is known to be true. Similarly we see from (2) that the line (3) passes through R. This line is therefore the required chord of contact. Hence:

If the point P: (x_1, y_1) is outside the parabola

$$y^2 = 4\,ax,$$

the equation

$$(3) \qquad \boldsymbol{y_1y = 2\,ax + 2\,ax_1}$$

represents the chord of contact corresponding to P.

105. Loci by elimination of parameters. A type of locus problem superficially, though not essentially, different from the type discussed in § 28 is illustrated by the following

Example: Find the locus of the midpoints of chords drawn through a fixed point on a circle.

Take the equation of the circle as

$$x^2 + y^2 = a^2,$$

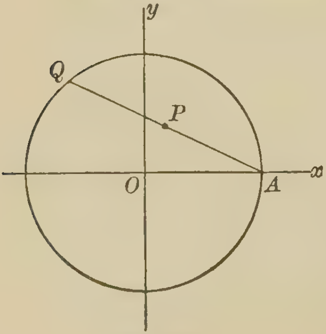


FIG. 92

and the point on the circle as $A: (a, 0)$. Suppose any one of the chords to intersect the circle at a point Q, whose coördinates may be denoted by (x_Q, y_Q), and let $P: (x, y)$ be the midpoint of AQ, so that P is a general point on the locus. The coördinates of Q are of course unknown, and in fact are variables. Our problem is to find an equation involving x and y but not x_Q or y_Q.

Since P is the midpoint of AQ, we have

$$x = \tfrac{1}{2}(a + x_Q), \tag{1}$$

$$y = \tfrac{1}{2} y_Q. \tag{2}$$

The fact that Q lies on the circle gives

$$x_Q^2 + y_Q^2 = a^2. \tag{3}$$

By (1) and (2),

$$x_Q = 2x - a,$$

$$y_Q = 2y;$$

substituting in (3), we find the equation of the locus to be

$$(2x - a)^2 + 4y^2 = a^2,$$

or

$$(x - \tfrac{1}{2}a)^2 + y^2 = \tfrac{1}{4}a^2:$$

a circle of radius $\frac{1}{2}a$ with center at $(\frac{1}{2}a, 0)$.

The type of problem above referred to is thus seen to be one in which it is difficult or impossible to obtain the equation of the locus directly, but where it is easy to obtain a number of equations involving x, y, and one or more auxiliary variables, called *parameters*. If then the parameters be eliminated, the result is the ordinary equation of the locus.

It will be recalled from algebra that to eliminate n variables, $n + 1$ equations involving them are required. Thus in this type of problem the number of equations that are obtained involving the parameters must be one greater than the number of parameters: in the above example three equations were obtained involving the two parameters x_Q, y_Q.

EXERCISES

1. From the theory of § 104, devise a new method for finding the equations of tangents to the parabola $y^2 = 4ax$ through an external point. Apply this method to find the tangents to the parabola $y^2 + 8x = 0$ through $(3, 1)$.

2. Find the tangents to the parabola $3y^2 = 5x$ through $(-3, 2)$.

3. Find the tangents to the parabola $2y^2 = x$ through $(1, \frac{3}{2})$.

4. Find the tangents to the parabola $3y^2 + 4x = 0$ through $(0, \frac{1}{4})$.

5. Solve Ex. 4 by means of the quadratic discriminant.

6. Show that the chord of contact corresponding to any point on the directrix passes through the focus.

7. State and prove the converse of the theorem of Ex. 6.

8. Prove that if a point lies on the latus rectum (produced), its chord of contact passes through the point of intersection of the axis and the directrix.

9. The points P_1, P_2 being external to a parabola, prove that if P_1 lies on the chord of contact corresponding to P_2, then P_2 lies on the chord corresponding to P_1.

10. From the theorem of Ex. 9 devise a method for constructing the tangents to a parabola through an external point.

11. Show that the x- and y- intercepts of the chord of contact corresponding to (x_1, y_1) with reference to the standard parabola are $-x_1$ and $\frac{2ax_1}{y_1}$ respectively. Hence devise a second construction for the tangents through an external point.

12. Draw the tangents to a parabola through any point on the axis.

13. Using the chord of contact, prove the theorem of § 102.

14. Find the locus of the midpoints of the ordinates of the circle $x^2 + y^2 = a^2$. *Ans.* $x^2 + 4y^2 = a^2$.

15. Find the locus of the midpoints of the ordinates of the parabola $y^2 = 4ax$.

16. Perpendiculars are dropped from the points of the parabola $y^2 = 4ax$ to the tangent at the vertex. Find the locus of the midpoints.

17. Solve Ex. 16 if the perpendiculars are dropped to the directrix.

18. Find the locus of the midpoints of the focal radii of a parabola (lines from the focus to the points of the curve). *Ans.* $y^2 = 2ax - a^2$.

19. Solve Ex. 18, using polar coördinates.

20. Find the locus of the midpoints of the focal chords of a parabola.

21. Find the locus of the point of intersection of perpendicular tangents to a parabola. *Ans.* The directrix.

22. Two tangents to a parabola make complementary angles with the axis (i.e. $mm' = 1$). Find the locus of their point of intersection. *Ans.* The latus rectum.

106. Diameter of a conic. A diameter of a circle has been defined in § 45 as a line through the center. However, since every line through the center bisects a certain system of parallel chords, and no other line does so, we might equally well define a diameter as the locus of the midpoints of a set of parallel chords.

In introducing the term with reference to the conics, the latter definition is adopted: a *diameter* of a conic section is *the locus of the midpoints of a system of parallel chords.*

It is not obvious from this definition that a diameter of a conic is a straight line; it will be shown, however, that such is the case. Further, it will appear that, for the ellipse and hyperbola, the definition of diameter as a straight line through the center would be equivalent to the definition just given.

107. Diameter of a parabola. To investigate those properties of the parabola which involve diameters, we will assume the equation of the parabola in the form

$$(1) \qquad y^2 = 4ax.$$

Let

$$y = mx + k,$$

where m is given and k is arbitrary, denote a family of parallel chords of the parabola. To find the diameter bisecting this system of chords, we must obtain an equation satisfied by the coördinates of the midpoint of any of the chords: i.e. an equation involving a and m but not involving k.

Let P_1: (x_1, y_1) and P_2: (x_2, y_2) be the endpoints of any chord

$$y = mx + k, \tag{2}$$

and let P: (x, y) be the midpoint, so that P is a general point on the locus. Now since P lies on the chord, we have in (2) an equation involving the coördinates of P, but also containing the variable parameter k. According to the method of § 105, if we can find a second equation involving x, y, and k, the result can be obtained by elimination of k.

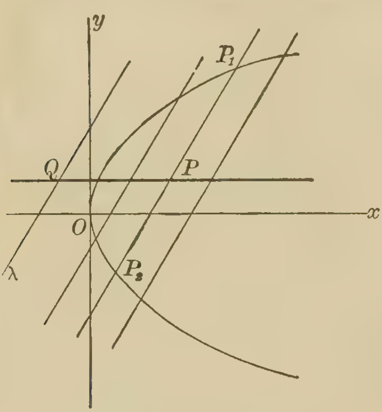


FIG. 93

Let us proceed to solve (1) and (2) as simultaneous equations. Substituting for y[*] in (1) from (2) we get

$$m^2x^2 + (2\,km - 4\,a)\,x + k^2 = 0.$$

Now the roots of this quadratic in x are the abscissas x_1, x_2 of the endpoints P_1, P_2. By § 49, their sum is the coefficient of x with the sign changed, when the coefficient of x^2 is made unity: i.e.

$$x_1 + x_2 = -\frac{2\,km - 4\,a}{m^2} = \frac{4\,a - 2\,km}{m^2}.$$

But since P is the midpoint of P_1P_2, its abscissa is

$$x = \tfrac{1}{2}(x_1 + x_2),$$

so that

$$x = \frac{2\,a - km}{m^2}. \tag{3}$$

---

[*] The result may be obtained somewhat more easily in this case by substituting for x, but the alternative adopted here illustrates better the general method.

Substituting the value of k from (2) in (3), we find

$$x = \frac{2a + m^2x - my}{m^2},$$

which reduces to

$$y = \frac{2a}{m}. \tag{4}$$

This is the equation of a straight line parallel to Ox.

We have thus proved * the

THEOREM: *Every diameter of a parabola is a straight line parallel to the axis of the curve.*

It will be remembered that a diameter of a circle may mean either a straight line through the center, or the segment of such a line lying inside the curve. Similarly, a diameter of a parabola is either the entire straight line parallel to the axis, or the half-line inside the curve. From the geometric point of view, only the latter meaning is allowable, since the midpoint of any chord must be a point inside the parabola, yet analytically the external points of the diameter may be shown to satisfy the definition equally well. Let us write the equation of the line λ (Fig. 93), and find the points of intersection of this "chord" with the curve. These points are of course imaginary, but if the point midway between them be obtained by the formulas of § 8, it will be found that this point (the point Q in the figure) has real coordinates and lies on the diameter bisecting the chords parallel to λ.

108. Geometric properties involving diameters. Two important properties of the parabola involving diameters are embodied in the following theorems:

THEOREM I: *The tangent at the end of a diameter is parallel to the bisected chords.*

THEOREM II: *The tangents at the ends of any chord intersect on the diameter bisecting that chord.*

---

* The proof is not quite complete. See Ex. 2 below.

It will be shown later that these theorems are true for the ellipse and hyperbola as well.

Theorem I may be proved by the student. To establish II, let us take the equation of the parabola as

$$y^2 = 4\,ax,$$

and assume the equation of the given chord in the form

$$(1) \qquad y_1 y = 2\,ax + 2\,ax_1.$$

Then, by § 104, the tangents at the ends of this chord intersect in the point (x_1, y_1). Now the slope of the line (1) is

$$m = \frac{2\,a}{y_1},$$

so that by (4), § 107, the equation of the corresponding diameter is

$$y = \frac{2\,a}{\frac{2\,a}{y_1}},$$

or

$$y = y_1.$$

Since this line passes through (x_1, y_1), the theorem is proved.

EXERCISES

1. Derive the result of § 107 by substituting the value of x from (2) in (1).

2. Show that the theorem of § 107 is incompletely proved, in that the chords $x = k$ are not taken care of, and carry out the proof for this case.

Derive the equations of the following diameters.

3. Bisecting those chords of the parabola $x^2 = 2\,y$ which are parallel to the line $y = 3\,x + 4$. *Ans.* $x = 3$.

4. Bisecting the chords $y = 2\,x + k$ of the parabola $y^2 - 3\,x + 5\,y = 6$.

5. Bisecting those chords of the parabola $2\,x^2 - 3\,x + 6\,y = 10$ that are perpendicular to the line $3\,x + 2\,y = 10$.

6. Bisecting chords of slope m of the parabola $x^2 = 4\,ay$.

7. Bisecting chords of slope 3 of the parabola $4\,x^2 - 4\,xy + y^2 + 2\,x - 2\,y = 0$. *Ans.* $y = 2\,x + 2$.

8. Bisecting the chords of slope $-\frac{1}{2}$ in the parabola $3\,x^2 - 6\,xy + 3\,y^2 + 4\,x = 0$.

9. Bisecting those chords of the parabola $(x - y)^2 = 3\,x$ that are parallel to Ox. *Ans.* $2\,x - 2\,y = 3$.

10. By the method of § 107, find the diameter of the circle $x^2 + y^2 = 4\,x$ bisecting the chords of slope $-\frac{1}{3}$. Check the result.

11. Prove analytically that, in the case of the circle, the two definitions of diameter mentioned in § 106 are equivalent.

12. Prove analytically that every diameter of a circle is perpendicular to the bisected chords.

13. Prove that with one exception no diameter of a parabola is perpendicular to the bisected chords. What is the exception?

14. State and prove the converse of the theorem of § 107.

15. Give an analytic proof of Theorem I, § 108.

16. Given a parabola, find its vertex geometrically.

17. Draw the tangent to a parabola parallel to a given line.

18. Draw the tangents to a parabola through a given external point.

19. Given a parabola and a point inside, draw the chord that is bisected at that point.

20. Find that chord of the parabola $y^2 = 2\,x$ that is bisected at (3, 1). *Ans.* $x - y = 2$.

21. Find that chord of the parabola $y^2 + 4\,x - 4\,y = 4$ that is bisected at (0, 3). *Ans.* $2\,x + y = 3$.

22. In the parabola $9\,x^2 - 24\,xy + 16\,y^2 - 18\,x - 50\,y + 19 = 0$, find the chord that is bisected at (1, 2). *Ans.* $24\,x + 5\,y = 34$.

23. Get the points of intersection of the line $y = x + 2$ with the parabola $y^2 = 4\,x$, and show that the point midway between these points of intersection lies on the diameter bisecting chords parallel to the given line. (Cf. the last paragraph of § 107.) Draw the figure.

24. Given two tangents to a parabola, with their points of contact, construct the curve.

25. Given two points and the tangent at one of them, together with the direction of the axis, construct the parabola.

26. Show that the chord of contact corresponding to any point on the directrix is perpendicular to the line joining that point to the focus.

27. Prove that the tangent at the vertex touches the circle drawn on any focal radius as a diameter.

28. Solve Ex. 27, using polar coördinates.

29. Given the focus and two points of a parabola, construct the curve, using the theorem of Ex. 27.

30. Given the axis and two points of a parabola, construct the curve.

[illegible]

CHAPTER IX

THE CENTRAL CONICS

109. Introduction. Since the ellipse and hyperbola have a center while the parabola does not, the former curves are called the *central conics*.

The central conics are so closely related, many theorems and methods being nearly or quite identical for the two curves, that it seems best to develop their properties simultaneously; this will be done in the present chapter. As in the case of the parabola, we begin with a set of exercises based on the theory already developed.

EXERCISES

Find the equations of the following conics, assuming in each case that the axes lie in the coördinate axes.

1. An ellipse for which the conjugate axis is 8 and the distance between the foci is 6.

2. An ellipse for which the latus rectum is 2 and the distance between the foci is $2\sqrt{2}$. *Ans.* $\frac{x^2}{4} + \frac{y^2}{2} = 1$.

3. A hyperbola with latus rectum 18 and distance between the directrices 3. *Ans.* $\frac{x^2}{9} - \frac{y^2}{27} = 1$.

4. A conic having the eccentricity $\frac{1}{2}$ and distance between the foci 5.

5. A conic having eccentricity $\sqrt{2}$ and latus rectum 6.

6. A conic having the distance between foci 6 and distance between directrices 4.

7. An ellipse for which the distance between the foci is $4\sqrt{6}$ and the rectangle on the axes is of area 20.

8. A conic with eccentricity $\frac{2}{3}$ and latus rectum $\frac{1}{3}$.

9. A conic through the points (2, 3), (4, 1).

10. An ellipse passing through (2, 1) and having the points $(\pm \sqrt{3}, 0)$ as foci.

11. A hyperbola with latus rectum 4 and slope of asymptotes $\pm \frac{1}{3}$.

12. A hyperbola with foci $(\pm 2, 0)$ and slope of asymptotes ± 3.

13. Show that a conic is determined if the foci and the length of either axis are given.

14. Show that if the directions of the axes are given a central conic may be made to satisfy four additional conditions.

15. Show that if the center is given a conic may be made to satisfy three additional conditions.

16. Show that if a focus and the corresponding directrix are given a central conic may be made to satisfy one additional condition.

17. Prove that if two conics have the same eccentricity they are similar. (Cf. § 96.)

18. Derive the polar equation of a conic with the center as pole, taking the focus as the point $(ae, 0)$, and the directrix as the line through $\left(\frac{a}{e}, 0\right)$ perpendicular to the initial line.

Ans. $r^2 (1 - e^2 \cos^2 \theta) = a^2 (1 - e^2)$.

19. Prove that if two similar conics are concentric with their transverse axes lying in the same line, they are similarly placed.

20. Find the locus of the midpoints of the abscissas of an ellipse.

21. Chords are drawn through the positive end of the major axis of the standard ellipse. Find the locus of their midpoints.

Ans. $\frac{4(x - \frac{1}{2}a)^2}{a^2} + \frac{4y^2}{b^2} = 1$.

22. Find the locus of the midpoints of the ordinates of a hyperbola.

23. Find the locus of the midpoints of the focal radii of a central conic. (Use polar cöordinates.)

24. Given a conic with its center, construct the axes.

25. Given an ellipse with its axes, construct the foci.

26. Given a hyperbola with its asymptotes, find the foci.

27. Given an ellipse with its axes, find the directrices.

28. Given a hyperbola with its axes, find the directrices.

29. Prove that the conjugate axis of a central conic is a mean proportional between the transverse axis and the latus rectum.

30. From the theorem of Ex. 29 derive a new solution for Exs. 25, 26.

31. A line segment of fixed length moves with its ends following two perpendicular lines. The line is divided by a point P into two segments of lengths a, b. Find the locus of P. *Ans.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

110. A new definition of the ellipse. Instead of the definition of the ellipse laid down in Chapter VII, the following is often used:

An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is constant. The fixed points are the foci and the constant sum is the transverse axis.

To show that this definition is equivalent to the one already given, we must prove that, for the ellipse as formerly defined, the sum of the focal distances of any point on the curve is constant, and that this property is possessed by no other curve.

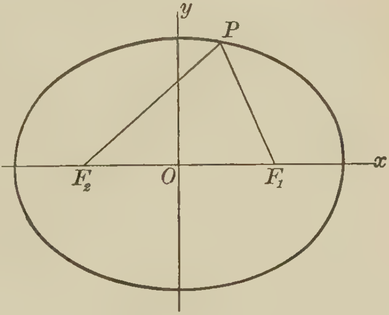


FIG. 94

Let P: (x, y) be any point on the ellipse

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

The distance of P from the focus F_1: $(ae, 0)$ is

$$F_1P = \sqrt{(x - ae)^2 + y^2}$$

$$(2) \qquad = \sqrt{x^2 - 2\,aex + a^2e^2 + y^2}.$$

By (1),

$$y^2 = a^2 - a^2e^2 - x^2 + e^2x^2.$$

Thus (2) takes the form

$$F_1P = \sqrt{a^2 - 2\,aex + e^2x^2},$$

or *

$$F_1P = a - ex. \tag{3}$$

Similarly,

$$F_2P = a + ex. \tag{4}$$

Addition of (3) and (4) gives

$$F_1P + F_2P = 2\,a.$$

Conversely, if a point moves so that the sum of its distances from two fixed points is constant, its locus is an ellipse. Let the moving point be $P: (x, y)$, the fixed points $F_1: (ae, 0)$, $F_2: (-\,ae, 0)$, and denote the constant sum by $2\,a$. Then

$$F_1P + F_2P = 2\,a,$$

or

$$\sqrt{(x - ae)^2 + y^2} + \sqrt{(x + ae)^2 + y^2} = 2\,a. \tag{5}$$

This equation can be rationalized by squaring twice† (the details are left to the student); the result is

$$x^2\,(1 - e^2) + y^2 = a^2\,(1 - e^2),$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2\,(1 - e^2)} = 1.$$

This completes the proof.

---

* We have

$$\sqrt{a^2 - 2\,aex + e^2x^2} = a - ex$$

rather than

$$\sqrt{a^2 - 2\,aex + e^2x^2} = ex - a$$

because for every point of the ellipse

$$x < \frac{a}{e},$$

so that

$$ex < a.$$

† Before squaring the first time, it is advisable to transpose one radical to the right member.

111. A new definition of the hyperbola. A characteristic property of the hyperbola analogous to the one just proved for the ellipse leads to the following definition:

A hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is constant. The fixed points are the foci and the constant difference the transverse axis.

To justify this definition, we proceed exactly as in the preceding article. The details may be carried out by the student.

112. Geometrical construction of the central conics. It follows from the results of §§ 110–111 that a central conic is determined if the foci and the transverse axis are given. Any number of points of the curve may then be constructed geometrically and a smooth curve drawn through them.

In many geometric problems involving the ellipse and hyperbola it is necessary to assume that the continuous curve can actually be drawn. To the list of fundamental assumptions laid down in § 97 we therefore add the following:

A central conic can be drawn if the foci and the transverse axis are given.

EXERCISES

1. Rationalize equation (5), § 110.

2. Prove that if a point describes a hyperbola, the difference of its distances from the foci is constant.

3. Prove the converse of the theorem of Ex. 2.

4. Given the foci and transverse axis of an ellipse, show how to construct points of the curve geometrically.

5. Given the foci and transverse axis of a hyperbola, show how to construct points of the curve.

6. Given the conjugate axis and one point of an ellipse, construct the curve (i.e. find the transverse axis and the foci). Cf. example (*a*), § 97.

7. Given the conjugate axis and one point of a hyperbola, construct the curve.

8. Given the transverse axis and one point of an ellipse, construct the curve.

9. Given the transverse axis and one point of a hyperbola, construct the curve.

Find the equations of the following loci.

10. A point moves so that the sum of its distances from $(0, 3)$, $(0, -3)$ is 8. Solve in two ways.

11. A point moves so that the difference of its distances from $(4, 2)$, $(4, -6)$ is 4.

12. A point moves so that the difference of its distances from $(-3, 0)$, $(-3, -8)$ is 1.

13. A point moves so that the difference of its distances from $(3, 6)$, $(-5, 2)$ is 2. Prove by the test for species of a conic that the locus is a hyperbola. *Ans.* $15x^2 + 16xy + 3y^2 - 34x - 8y - 20 = 0$.

14. A point moves so that the sum of its distances from $(0, 0)$, $(4, 2)$ is 10. *Ans.* $21x^2 - 4xy + 24y^2 - 80x - 40y - 400 = 0$.

15. Prove that if the foci of a central conic are placed symmetrically with respect to the origin, the equation of the conic can contain no terms of the first degree.

16. Given the foci and one point of an ellipse, find the axes.

17. Given the foci and one point of a hyperbola, find the axes.

18. Given a focus, the corresponding directrix, and one point of a conic, construct the curve.

19. Given a focus, the corresponding directrix, and the eccentricity of a conic, draw the curve.

20. Given the foci and asymptotes of a hyperbola, find the vertices.

21. Given a hyperbola with its center, draw the asymptotes.

22. Given the center, directrices, and eccentricity of a conic, construct the curve.

23. A circle passes through a given point and touches a given circle. Construct the locus of its center. Consider the cases in which the given point is outside, on, inside, and at the center of the given circle.

24. Draw a circle of given radius passing through a given point and touching a given circle. Solve by two methods. Consider various cases.

25. Draw a circle passing through two given points and touching a given circle.

26. A circle touches two fixed circles. Construct the locus of its center, for all cases.

27. A circle touches the circles $x^2 + y^2 = 4$, $x^2 + y^2 - 12x = 64$. Find the equation of the locus of its center, and draw the figure.

Ans. $3x^2 + 4y^2 - 18x = 81$.

28. The sound of a gun and the ring of the ball on the target are heard simultaneously at a point P. Construct the locus of P.

113. Tangent at a given point of contact. The method of § 98 may of course be applied to find a tangent to any central conic when the point of contact is given.

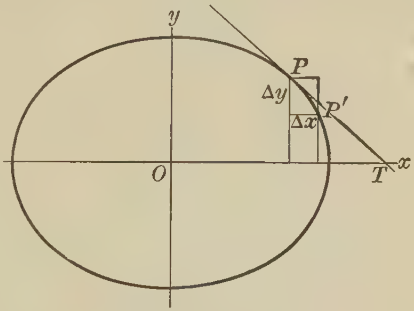


FIG. 95

To find the tangent to the standard ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point P: (x_1, y_1) on the curve, we will first, for convenience, clear the equation of fractions:

$$b^2x^2 + a^2y^2 = a^2b^2. \tag{1}$$

Choosing a neighboring point P': $(x_1 + \Delta x, y_1 + \Delta y)$ on the curve, we substitute its coördinates in (1):

$$b^2(x_1 + \Delta x)^2 + a^2(y_1 + \Delta y)^2 = a^2b^2,$$

or

$$b^2x_1^2 + 2b^2x_1\,\Delta x + b^2\,\overline{\Delta x}^2 + a^2y_1^2 + 2a^2y_1\,\Delta y + a^2\,\overline{\Delta y}^2 = a^2b^2. \tag{2}$$

Since the point (x_1, y_1) lies on the curve (1), we have

$$b^2x_1^2 + a^2y_1^2 = a^2b^2, \tag{3}$$

so that (2) reduces to

$$2b^2x_1\,\Delta x + b^2\,\overline{\Delta x}^2 + 2a^2y_1\,\Delta y + a^2\,\overline{\Delta y}^2 = 0.$$

Now divide through by Δx:

$$2b^2x_1 + b^2\,\Delta x + 2a^2y_1\frac{\Delta y}{\Delta x} + a^2\frac{\Delta y}{\Delta x}\Delta y = 0.$$

Letting Δx approach the limit 0, we get the relation

$$2\,b^2x_1 + 2\,a^2y_1\,m = 0,$$

where m is the slope of the curve at (x_1, y_1); hence

$$m = -\frac{b^2x_1}{a^2y_1}. \tag{4}$$

The equation of the required tangent is therefore

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1),$$

or, after clearing of fractions and transposing,

$$b^2x_1x + a^2y_1y = b^2x_1^2 + a^2y_1^2.$$

By (3), this becomes

$$b^2x_1x + a^2y_1y = a^2b^2;$$

dividing by a^2b^2, we get the form

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1. \tag{1}$$

The corresponding result for the hyperbola is evidently

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1. \tag{2}$$

114. A property of the tangent to a central conic. By (4), § 113, the equation of the normal to the standard ellipse at a point P: (x_1, y_1) is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

The x-intercept of this line is evidently

$$ON = \frac{a^2 - b^2}{a^2}x_1 = e^2x_1.$$

Therefore, in Fig. 96, we have

$$NF_1 = ae - e^2x_1 = e\,(a - ex_1),$$
$$F_2N = ae + e^2x_1 = e\,(a + ex_1),$$

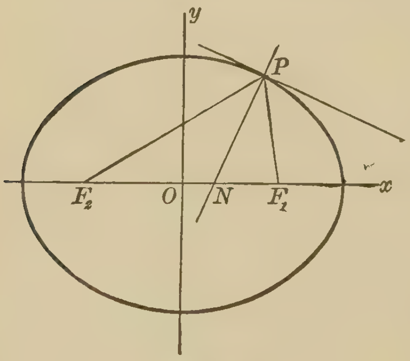


FIG. 96

whence

$$\frac{NF_1}{F_2N} = \frac{a - ex_1}{a + ex_1},$$

or by (3) and (4) of § 110,

$$\frac{NF_1}{F_2N} = \frac{F_1P}{F_2P}.$$

Thus the normal PN bisects the angle between the focal radii drawn to P. Since the tangent is perpendicular to the normal, we have proved

THEOREM I: *The tangent to an ellipse bisects externally the angle between the focal radii drawn to the point of contact.*

The property of the hyperbola analogous to the one just proved for the ellipse is expressed by

THEOREM II: *The tangent to a hyperbola bisects internally the angle between the focal radii drawn to the point of contact.*

That is, in Fig. 97, the line TP bisects the angle F_1PF_2.

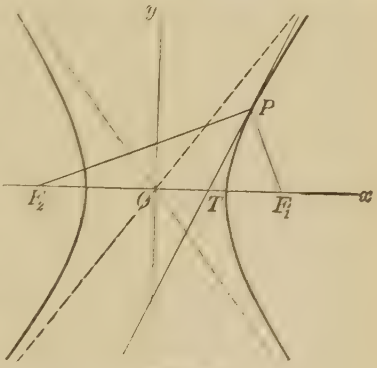


FIG. 97

The proof is similar to that for the ellipse and may be carried out by the student.

115. Tangents having a given slope. By use of the quadratic discriminant according to a method now familiar, we may determine tangents of given slope to any central conic.

In particular, it is easy to show that the line

$$y = mx + k$$

is tangent to the standard ellipse if

$$k = \pm \sqrt{a^2m^2 + b^2};$$

i.e. *the lines*

$$(3) \qquad y = mx \pm \sqrt{a^2m^2 + b^2}$$

are tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for all values of m.

Similarly, *the lines*

$$y = mx \pm \sqrt{a^2m^2 - b^2} \tag{4}$$

are tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for all values of m.

It should be noted, however, that the lines (4) are imaginary when $a^2m^2 < b^2$, i.e. when $-\frac{b}{a} < m < \frac{b}{a}$.

EXERCISES

Find the equations of the following tangents.

1. To the ellipse $4x^2 + y^2 = 8$ at $(1, -2)$.

2. To the ellipse $x^2 + 3y^2 - 2x - 6y = 9$ at $(0, 3)$.
Ans. $x - 6y + 18 = 0$.

3. To the hyperbola $2x^2 - y^2 + 4 = 0$ at $(4, 6)$.

4. To the hyperbola $x^2 - 4y^2 - 5x - 6y = 0$ at $(0, 0)$.
Ans. $5x + 6y = 0$.

5. To the conic $xy = 3$ at (x_1, y_1). *Ans.* $y_1x + x_1y = 6$.

6. To the conic $x^2 - 3xy + 3y^2 + x - 2y - 2 = 0$ at $(1, 0)$.
Ans. $5y = 3x - 3$.

7. To the conic $3x^2 + 2xy - 2x = 3$ at $(1, 1)$.

8. To the curve $x^2 + 2xy - 2y^2 - 3x = 0$ at $(2, 1)$.

9. Prove that if a conic passes through the origin, the equation of the tangent at that point is found by equating to 0 the group of terms of the first degree in the equation of the conic.

10. Find the angle at which the conics $x^2 + 2y^2 - 20y + 12 = 0$, $x^2 = 6y$ intersect, and draw the figure.
Ans. $\phi = 45°$; $\phi = \arctan \frac{1}{6}\sqrt{6}$.

11. Given a conic with its center, draw the tangent and normal at any point.

12. Prove Theorem II of § 114.

13. Given one focus of an ellipse, and two tangents with their points of contact, construct the curve.

14. Given one focus of a hyperbola, and two tangents with their points of contact, construct the curve.

15. Prove that the perpendicular from the focus upon any tangent to an ellipse, and the line joining the center to the point of contact, intersect in a point on the directrix.

16. Prove that the perpendicular from the focus upon any tangent to a hyperbola, and the line joining the center to the point of contact, intersect in a point on the directrix.

Find the equations of the following tangents.

17. To the hyperbola $3x^2 - 2y^2 + 5 = 0$ perpendicular to the line $4x + 2y = 3$.

18. To the ellipse $x^2 + 4y^2 + 4x + 6y = 6$ parallel to the line $y = x$.

19. To the conic $3x^2 + y^2 = x$ parallel to the line $2y = 3x$.

20. To the conic $3x^2 - xy + y^2 + x - 3y = 0$ perpendicular to the line $3x + y + 2 = 0$.

21. To the conic $x^2 + xy + y^2 + x = 2$ perpendicular to Ox.

22. To the curve $x^2 - y^2 - 2x - 2y + 4 = 0$ through $(3, 1)$.

23. To the curve $xy = 2$ through $(6, -1)$.

24. To the ellipse $2x^2 + y^2 = 18$ through $(3, 3)$.

25. To the hyperbola $3xy - 4y^2 + 2x = 0$ through $(0, 2)$.

26. Derive formula (**3**) of § 115.

27. Prove that the quantity $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ is positive, zero, or negative according as the point (x_1, y_1) is outside, on, or inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Cf. § 102.)

28. Prove that two tangents can be drawn to an ellipse through an external point. (See Ex. 27.)

29. Prove that the quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is negative, zero, or positive according as (x_1, y_1) is outside, on, or inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

30. Prove that two tangents can be drawn to a hyperbola through an external point. (See Ex. 29.)

31. Prove that two tangents can be drawn to an ellipse in any direction.

32. Prove that two tangents, or none, can be drawn to a hyperbola in a given direction, according as the line through the center in that direction does not, or does, intersect the curve. Is there any exception?

33. Prove that the product of the distances of the foci from any tangent to an ellipse is constant (equal to b^2).

34. Prove that the product of the distances of the foci from any tangent to a hyperbola is constant (equal to b^2).

35. Given the foci and any tangent to a central conic, construct the curve. (See Exs. 33, 34.)

116. Another property involving tangents. In Fig. 95, p. 179, the x-intercept of the tangent

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \tag{1}$$

to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$OT = \frac{a^2}{x_1}. \tag{2}$$

Since the fact that b is less than a is not involved in the derivation of (1), it follows that that formula, and likewise the result (2), would hold even if a were used to denote the semi-minor axis.

Since, by (2), the x-intercept is independent of b, we deduce at once

THEOREM I: *If two ellipses have one axis in common*, the tangents at points whose distances from the other axis are equal intersect the common axis (produced) at the same point.*

---

* Either the major axes, the minor axes, or the major of one and the minor of the other.

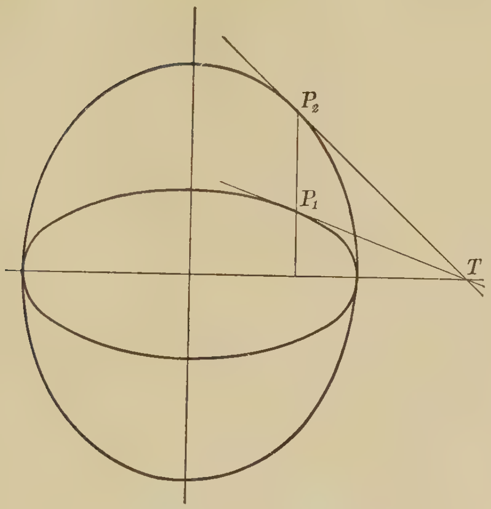


FIG. 98

That is, in Fig. 98, the tangents at P_1 and P_2 intersect the common axis (produced) at the same point T.

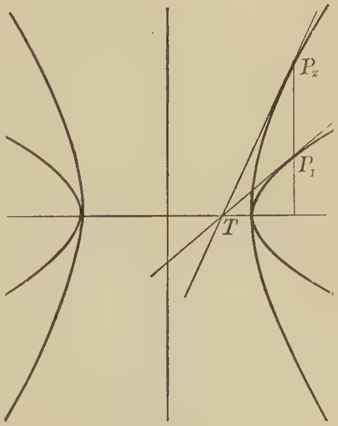


FIG. 99

The corresponding property of the hyperbola, the proof of which is left to the student, is as follows:

THEOREM II: *If two hyperbolas have a common transverse axis, the tangents at points whose distances from the conjugate axis are equal intersect the transverse axis at the same point.*

Thus in Fig. 99 the tangents at P_1 and P_2 intersect the transverse axis at the same point T.

117. Auxiliary circles; eccentric angle. The circle described on the major axis of an ellipse as a diameter is called the *major auxiliary circle;* that on the minor axis is the *minor auxiliary circle.* For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equations of these circles are obviously

$$x^2 + y^2 = a^2,$$

$$x^2 + y^2 = b^2$$

respectively.

If the ordinate MP of a point P on an ellipse be produced until it cuts the major auxiliary circle at a point Q, and a line OQ be drawn from the center to Q, the angle $\phi = \angle MOQ$ is called the *eccentric angle* of the point P.

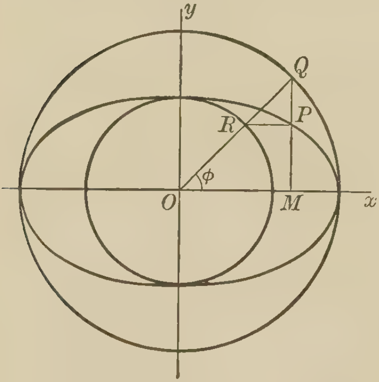


FIG. 100

Evidently, in Fig. 100,

$$OM = OQ \cos \phi;$$

that is,

$$x = a \cos \phi.$$

Substituting this value of x in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and solving for y^2, we get

$$y^2 = b^2 (1 - \cos^2 \phi),$$

or

$$y = b \sin \phi.$$

It follows from this that the line OQ and the line through P parallel to the major axis intersect the minor auxiliary circle at the same point R.

118. Parametric equations. Instead of deriving the properties of a curve from its ordinary cartesian equation, it is sometimes more convenient to represent the curve by means of two equations giving x and y respectively in terms of some third variable. (These two equations taken together must of course express some characteristic property of the curve.) This third variable is called a *parameter*, and the two equations are *parametric equations* of the curve. By using different auxiliary variables a curve may be thus represented in a variety of ways. To obtain the ordinary cartesian equation it is only necessary to eliminate the parameter between the two equations.

A curve may be represented by parametric equations in less simple ways. For instance, each of the equations may involve both x and y as well as the parameter. Again, three equations may be given, two of which express x and y respectively in terms of two parameters, while the third gives a relation between the parameters.

Various examples of parametric representation have already occurred in our work. In § 117, the equations

$$x = a \cos \phi,$$
$$y = b \sin \phi$$

are parametric equations of the ellipse. In § 107, the equations

$$y = mx + k,$$
$$x = \frac{2a - km}{m^2}$$

represent the diameter of the parabola in terms of the parameter k. And in the example of § 105 the equations

$$x = \tfrac{1}{2}(a + x_Q),$$
$$y = \tfrac{1}{2} y_Q,$$
$$x_Q^2 + y_Q^2 = a^2$$

are the equations of the required locus in terms of the parameters x_Q, y_Q.

119. Parametric equations of the ellipse. As already noted, the equations*

$$(5)\qquad \begin{cases} x = a\cos\phi, \\ y = b\sin\phi \end{cases}$$

obtained in § 117 are the *parametric equations of the ellipse* in terms of the eccentric angle. For, we have proved in that article that the coördinates of every point of the ellipse satisfy equations (5); it remains to prove, conversely, that every point whose coördinates satisfy (5) lies on the ellipse, and this can be done merely by eliminating the parameter. If we write (5) in the form

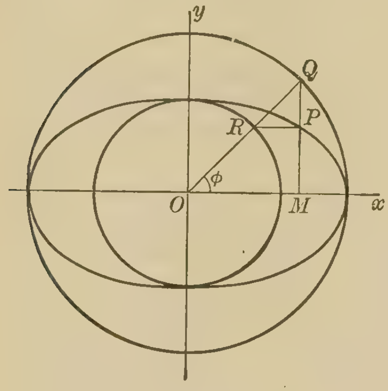


Fig. 100

$$\frac{x}{a} = \cos\phi, \quad \frac{y}{b} = \sin\phi,$$

square both members of each equation, and add, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi + \sin^2\phi = 1.$$

120. Parametric equations of the hyperbola. The standard hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

may be represented by the parametric equations

$$(6)\qquad \begin{cases} x = a\sec\phi, \\ y = b\tan\phi. \end{cases}$$

---

* The student is warned against the rather common error of confusing the eccentric angle ϕ with the polar angle θ. The equations (5) do not involve polar coördinates in any way.

If we write these equations in the form

$$\frac{x}{a} = \sec \phi, \quad \frac{y}{b} = \tan \phi,$$

square both members of each, and subtract, the result is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 \phi - \tan^2 \phi = 1.$$

As in the case of the ellipse, the angle ϕ is called the *eccentric angle*, but the geometric meaning is quite different. The eccentric angle corresponding to any point P may be constructed as follows: Drop a perpendicular PM from P to the transverse axis. Draw the circle on the transverse axis as a diameter, and from M draw a tangent to this circle. Join Q, the point of contact, and O, the center. Then

$$\phi = \angle MOQ.$$

The proof is left to the student.

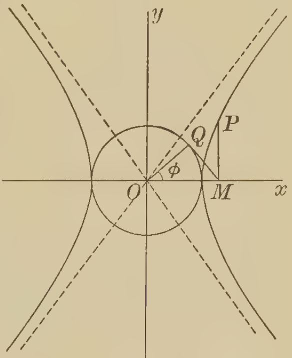


FIG. 101

EXERCISES

1. In the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$, find the points whose eccentric angles are (*a*) 30°, (*b*) 150°.

2. In the ellipse $4x^2 + 16y^2 = 1$, find the points whose eccentric angles are (*a*) 45°, (*b*) 240°.

3. In the hyperbola $x^2 - y^2 = 4$, find the points whose eccentric angles are (*a*) 60°, (*b*) 135°.

4. In the hyperbola $3x^2 - 4y^2 = 6$, find the points whose eccentric angles are (*a*) 0°, (*b*) 180 .

In each of the following cases, find the eccentric angle corresponding to the given point.

5. The point $(\frac{3}{2}, \frac{1}{2})$ on the conic $x^2 + 3y^2 = 3$.

6. The points $x = 2$ on the conic $x^2 + 4y^2 = 8$.

7. The point (4, 1) on the conic $x^2 - 4y^2 = 12$.

8. The point (2, − 3) on the conic $9x^2 - 2y^2 = 18$.

9. What curve is represented by $x = \frac{1}{4} a \tan^2 \phi$, $y = a \tan \phi$?

10. Given the curve of Ex. 9, construct the angle ϕ corresponding to any point.

Find the cartesian equations of the following curves.

11. $x = a \sin 2\phi$, $y = a \cos 2\phi$.

12. $x = 2t$, $y = t^2$.

13. $x = y + \sin \phi$, $y = \cos^2 \phi$.

14. $x = 2 \cos n\phi$, $y = 3 \sin n\phi$.

15. $x = 3 \csc \phi$, $y = 4 \cot \phi$.

16. $x = at$, $y = b\sqrt{1 - t^2}$.

17. $x = \frac{1}{2} \sin 2\alpha$, $y = \sin^2 \alpha$.

18. Prove the construction given in the last paragraph of § 120.

19. Solve Ex. 31, p. 175, by a new method.

20. The normal is drawn to an ellipse at a point P, and to the major auxiliary circle at the corresponding point Q (Fig. 100). Find the locus of the point of intersection of these normals.

Ans. $x^2 + y^2 = (a + b)^2$.

21. Give another construction for the eccentric angle of a point on an ellipse.

22. Give another construction for the eccentric angle of a point on a hyperbola.

23. Given the transverse axis and one point of an ellipse, construct the curve by a new method. (Ex. 8, p. 178.)

24. Given the transverse axis and one point of a hyperbola, construct the curve. (Ex. 9, p. 178.)

25. Given the center of an ellipse, the directions of the axes, and one point with its eccentric angle, construct the curve.

26. Prove Theorem II, § 116.

27. From Theorem I, § 116, derive a new construction for the tangent and normal at any point of an ellipse.

28. From equation (2) of § 116 derive a third construction for the tangent and normal at any point of an ellipse. Compare this construction with that of Ex. 27.

29. Derive a new construction for the tangent and normal at any point of a hyperbola.

30. Find the y-intercept of the tangent to the standard hyperbola; hence derive another construction for the tangent and normal at any point of the curve.

31. Given the center of an ellipse, the directions of the axes, and a tangent with its point of contact, construct the curve.

32. Prove that every tangent to a hyperbola passes between the vertices.

33. Given the transverse axis and a tangent to an ellipse, draw the curve.

34. Given the transverse axis and a tangent to a hyperbola, draw the curve.

121. Chord of contact. By precisely the same method that was used in §§ 56, 104, we may obtain the following result:

The equation of the chord of contact corresponding to the external point $P:(x_1, y_1)$, *with reference to the conic*

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

is

$$\frac{x_1x}{a^2} \pm \frac{y_1y}{b^2} = 1. \tag{7}$$

The proof is left to the student.

122. Diameters. To find the equation of the diameter bisecting chords of slope m of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

let us write the equation of the family of chords in the form

$$y = mx + k. \tag{2}$$

Substituting the value of y from (2) in (1), we get, after clearing of fractions,

$$(b^2 + a^2m^2)\, x^2 + 2\, a^2kmx + a^2k^2 - a^2b^2 = 0,$$

or

$$x^2 + \frac{2\, a^2km}{b^2 + a^2m^2}\, x + \frac{a^2k^2 - a^2b^2}{b^2 + a^2m^2} = 0.$$

The sum of the roots of this quadratic in x is

$$x_1 + x_2 = -\frac{2a^2km}{b^2 + a^2m^2}.$$

Therefore the abscissa x of the midpoint of any one of the chords (2) is

$$\begin{aligned} x &= \tfrac{1}{2}(x_1 + x_2) \\ &= -\frac{a^2km}{b^2 + a^2m^2}. \end{aligned} \qquad (3)$$

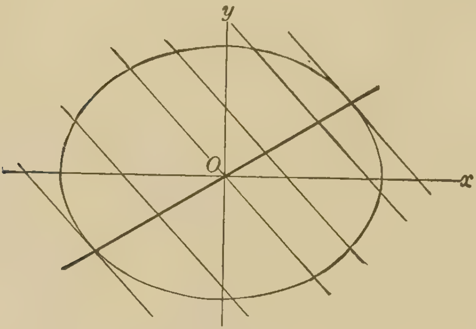


FIG. 102

Eliminating k by substituting from (2) in (3), we get the equation

$$x = -\frac{a^2m(y - mx)}{b^2 + a^2m^2},$$

which reduces to

$$y = -\frac{b^2}{a^2m}x. \qquad (8)$$

The corresponding result for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is easily seen to be

$$y = \frac{b^2}{a^2m}x. \qquad (9)$$

Thus we have the *

THEOREM: *Every diameter of a central conic is a straight line through the center.*

---

* The theorem is not yet proved for the diameter bisecting the chords $x = k$. See Ex. 20 below.

The converse of this theorem is true for the ellipse. For the hyperbola it is true if the asymptotes be excepted. Evidently no proper chords can be drawn parallel to an asymptote, since a line so placed intersects the curve in only one point, but if the lines parallel to an asymptote be thought of as a family of chords there are reasons*, both analytic and geometric, why the asymptote must be considered as a diameter *bisecting the chords parallel to itself.* With this convention the converse is true without exception.

123. Geometric properties involving diameters. Two noteworthy properties of the conics involving diameters are the following:

THEOREM I: *The tangents at the ends of any diameter*† *of a conic are parallel to the bisected chords.*

THEOREM II: *The tangents at the ends of any chord of a conic intersect on the diameter bisecting that chord.*

These theorems have already been proved for the parabola (§ 108). The proof of Theorem I for the central conics is left to the student.

To prove THEOREM II for the ellipse let us take the equation of the curve in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

that of the chord in the form

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

This is the chord of contact corresponding to the external point

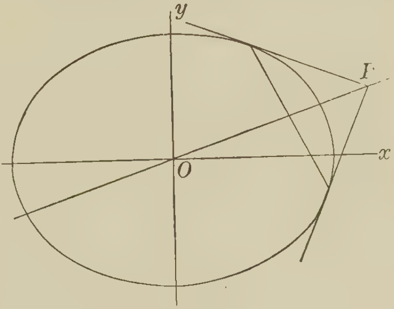


FIG. 103

---

* For example, see Ex. 21 below.

† A diameter of a hyperbola may not intersect the curve (see Ex. 24 below), in which case the theorem is of course inapplicable.

$P: (x_1, y_1)$, so that the tangents at the ends of this chord intersect at (x_1, y_1). The slope of the chord is

$$m = -\frac{b^2x_1}{a^2y_1};$$

substituting in (8) we obtain the equation of the corresponding diameter in the form

$$y = \frac{y_1}{x_1}x.$$

Since this line passes through (x_1, y_1), the theorem is proved.

The proof for the hyperbola is similar.

EXERCISES

1. Derive the result of § 121.

2. The points P_1, P_2 being external to a given central conic, prove that if P_1 lies on the chord of contact corresponding to P_2, then P_2 lies on the chord corresponding to P_1. (Cf. Ex. 9, p. 166.)

3. Draw the tangents to a central conic through an external point.

4. Show that the chord of contact corresponding to any point on the directrix passes through the focus. (Cf. Ex. 6, p. 166.)

5. Show that the chord of contact corresponding to any point on the directrix is perpendicular to the line joining that point to the focus. (Cf. Ex. 26, p. 172.)

6. Prove that if a point lies on the latus rectum (produced), its chord of contact passes through the point of intersection of the directrix and the axis. (Cf. Ex. 8, p. 166.)

7. Using the chord of contact, find the tangents through (4, 4) to the conic $3x^2 - 4y^2 = 8$. *Ans.* $3x - 2y = 4$, $9x - 10y + 4 = 0$.

8. Find the tangents to the ellipse $x^2 + 4y^2 = 4$ through (3, 1).

9. Find the tangents to the ellipse $10x^2 + 2y^2 = 7$ through $(-2, -1)$.

10. Find the tangents to the hyperbola $x^2 - y^2 = 1$ through $(0, -1)$.

Find the equations of the following diameters.

11. Of the ellipse $3x^2 + 2y^2 = 8$ bisecting the chords $3x + y = k$.

12. Of the hyperbola $x^2 - 2y^2 + 4 = 0$ bisecting chords parallel to the line $x + 2y = 3$.

13. Of the conic $2x^2 - xy = 3$ bisecting chords of slope 3.
Ans. $y = x$.

14. Of the conic $x^2 + 2y^2 + 7x + 3y - 5 = 0$ bisecting chords of slope -2.

15. Of the conic $3x^2 - 4xy + 2y^2 - x - 3y + 6 = 0$ bisecting chords parallel to the y-axis. *Ans.* $4x - 4y + 3 = 0$.

16. Of the conic $3x^2 - y^2 + 2x - 3y - 3 = 0$ bisecting chords parallel to the conjugate axis.

17. Given a conic, construct its center.

18. Draw the tangents to a central conic parallel to a given line.

19. Given a point inside a central conic, construct the chord that is bisected at that point.

20. Prove the theorem of § 122 for the chords $x = k$.

21. Show that the slope m' of any diameter of the standard hyperbola and the slope m of the bisected chords are connected by the relation $mm' = \frac{b^2}{a^2}$; hence that an asymptote must be considered as bisecting the chords parallel to itself.

22. Prove the converse of the theorem of § 122.

23. Prove Theorem I of § 123.

24. Prove that a diameter of a hyperbola does or does not intersect the curve according as that one of its chords that passes through the center does not or does intersect the curve.

25. Prove that in general a diameter is not perpendicular to its chords. What exceptions are there?

124. Points at infinity; curves intersecting at infinity. In Fig. 104, let P be a fixed point, λ a fixed line, and λ' a variable line through P intersecting λ in the point Q. If now λ' be taken more and more nearly parallel to λ, the point of intersection Q *recedes indefinitely from* P: this fact may be conveniently expressed by saying that *parallel lines intersect at infinity.*

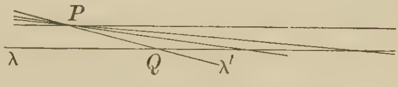


FIG. 104

Again, let a line be drawn through a fixed point P intersecting a hyperbola in two points. As the line becomes more nearly parallel to an asymptote, one point of intersection recedes indefinitely: thus we say that a line parallel to an asymptote has one finite intersection with the curve, while the other intersection lies at infinity. If the fixed point P lies on an asymptote, both points of intersection recede indefinitely: thus an asymptote intersects the hyperbola in two points at infinity.

The argument may be extended to curves of any degree. For instance, two conics .intersect in four points (real or imaginary), but in particular cases one or more of these points may lie at infinity.

The advantages arising from the introduction of the "point at infinity" are two-fold: it is very useful in the development of certain portions of our theory, as will appear in the next few articles, and it enables us to state without qualification various theorems that would otherwise be subject to exception. It must not be forgotten, however, that *the point at infinity is merely an ideal concept introduced for convenience;* points at infinity do not, of course, have any actual existence in the ordinary geometric sense.

125. Number of intersections at infinity. Given the equations of any two curves, the number of intersections at infinity can be determined by the theorem of § 44 on infinite roots of an algebraic equation. For instance, if the given curves are a straight line and a conic, let us substitute the value of y (say) from the linear equation into the one of second degree, thus obtaining a quadratic in x. If this quadratic actually contains the term in x^2, the curves intersect in two (real or imaginary) points. If the term in x^2 drops out, one root is infinite, and the line intersects the conic in one finite point, the other intersection lying at infinity. If the terms in x^2 and x both drop out, both roots are infinite, and both intersections lie at infinity.

Similar argument applies to any two curves. *A curve of the mth degree intersects a curve of the nth degree in mn points, real or imaginary.* If when the equations are solved as simultaneous the number of solutions obtained is less than mn, the intersections that are lacking lie at infinity.

Examples: (*a*) Prove analytically that if two lines are parallel, they intersect at infinity; and conversely.

Let

$$y = mx + b_1,$$
$$y = mx + b_2$$

be any two parallel lines. Eliminating y by subtraction, we get necessarily a linear equation in x, which here takes the form

$$b_1 - b_2 = 0.$$

Since the coefficient of x is 0, the root is infinite, and the theorem is proved.

Conversely, let

$$y = m_1x + b_1,$$
$$y = m_2x + b_2$$

be two lines known to intersect at infinity. Eliminating y by subtraction, we get

$$(m_1 - m_2)\,x + b_1 - b_2 = 0.$$

Since the lines intersect at infinity, this equation must have an infinite root: hence

$$m_1 = m_2.$$

(*b*) Find the equation of the straight line through (2, 3) and through the point at infinity on the line $x + y = 6$.

By example (*a*) (converse), the required line must be parallel to the given line, so that its equation is

$$x + y = 5.$$

126. Test for the species of a conic. We will now prove the theorem of § 87 that an equation of the second degree represents an ellipse, parabola, or hyperbola according as the quantity $B^2 - 4\,AC$ is negative, zero, or positive.

From the polar equation of a conic,

$$r = \frac{l}{1 + e \cos \theta},$$

it appears that when

$$1 + e \cos \theta = 0,$$

i.e. when

$$\cos \theta = -\frac{1}{e}, \tag{1}$$

the radius vector is infinite. Now (1) is satisfied by no (real) value of θ if $e < 1$, by one value if $e = 1$, and by two values if $e > 1$; hence a line through the focus meeting the conic in a point at infinity can be drawn *in no direction, in one direction, or in two directions according as the conic is an ellipse, parabola, or hyperbola.* It follows from example (*a*), § 125, that the same statement can be made regarding a line through any other point than the focus.

Therefore, given a conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2}$$

and a line through the origin

$$y = mx \tag{3}$$

(a line through any other point would do as well), let us substitute the value of y from (3) in (2). This gives, after collecting terms,

$$(A + Bm + Cm^2)\, x^2 + (D + Em)\, x + F = 0.$$

The line and the conic will intersect in a point at infinity if the coefficient of x^2 is 0: i.e. if

$$A + Bm + Cm^2 = 0.$$

The values of m determined by this quadratic will be imaginary, real and equal, or real and different according as the discriminant $B^2 - 4\,AC$ is negative, zero, or positive, and there will be, correspondingly, no direction, one direction, or two directions in which a line can be drawn meeting the curve at infinity. This completes the proof.

EXERCISES

Investigate the intersection of the following pairs of curves.

1. $x - 3y + 1 = 0$, $x^2 - 2xy - 3y^2 + 2x + 2y = 0$.
Ans. $(-\frac{1}{4}, \frac{1}{4})$, one at infinity.

2. $x^2 - 4y^2 - 2x - 8y = 7$, $x - 2y + 1 = 0$. Trace the curves.

3. $25x^2 - 10xy + y^2 + x - 3y = 6$, $y = 5x - 2$.

4. $x^2 - y^2 + 3x - y = 0$, $x^2 + xy + y = 0$.
Ans. (0, 0), two imaginary, one at infinity.

5. $xy - y^2 + 2x = 0$, $xy - y^2 + y = 0$.

6. $(y - k)^2 = 4ax$, $(y + k)^2 = 4ax$.

7. $y = x$, $x^3 - x^2y - y + 2 = 0$.

8. Prove that if a line is parallel to the axis of a parabola, it intersects the curve in one finite point and one at infinity.

9. Prove the converse of the theorem of Ex. 8.

10. Prove that no line can intersect a parabola in two points at infinity.

11. Prove that two parabolas having the same axis intersect in two points at infinity.

12. Prove that any two circles intersect in two points at infinity; hence that there are two points at infinity through which all circles pass. (Since the circle has no infinite branch, these points are necessarily imaginary; they are called the *circular points at infinity.*)

13. Prove that the ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$ intersect in four imaginary points at infinity.

14. Investigate the intersection of two concentric circles.

15. Show that the equation of any parabola can be written in the form $(ax + cy)^2 + dx + ey + f = 0$, and prove that the axis of this parabola is parallel to the line $ax + cy = 0$.

16. Find the slope of the axis of the parabola $4x^2 + 4xy + y^2 - 3x + 7y + 2 = 0$. *Ans.* -2.

17. Find the slope of the axis of the parabola $9x^2 - 30xy + 25y^2 - x + 3 = 0$.

Find the equations of the following parabolas by the method of § 93, using as the fourth point the point at infinity on the given line. Check the results.

18. Axis parallel to the line $y = 0$, passing through $(0, 0)$, $(1, 1)$, $(7, 3)$. *Ans.* $2y^2 - 3x + y = 0$.

19. Passing through $(-2, 2)$, $(4, 8)$, $(1, \frac{1}{2})$, with axis parallel to Oy.

20. Passing through $(0, 2)$, $(1, 4)$, $(2, 0)$, with axis parallel to the line $2x + y = 3$. *Ans.* $12x^2 + 12xy + 3y^2 - 44x - 26y + 40 = 0$.

21. Passing through $(-1, 3)$, $(0, 7)$, $(1, -1)$, with axis parallel to the line $4x + y = 6$.

127. Asymptotes. If the tangent to a curve approaches a definite limiting position as its point of contact recedes to infinity, the line so approached is called an *asymptote*. Thus an asymptote may be spoken of as a tangent whose point of contact lies at infinity. While the hyperbola is the only conic having asymptotes, many higher plane curves possess one or more of these lines, and they play an important part in the geometry of those curves.

128. Asymptotes of the hyperbola. It was shown in § 78 that the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

approaches more and more closely the lines

$$y = \pm \frac{b}{a} x.$$

It will now be proved that these lines are asymptotes * according to the definition of § 127.

The tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point $P: (x_1, y_1)$ is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1. \tag{1}$$

---

* It might seem that if a line and a curve approach each other indefinitely the line is necessarily an asymptote, but examples can be found among the higher plane curves in which this is not the case. The argument of the present article shows that such examples cannot occur among the conics.

If in (1) we let both x_1 and y_1 increase indefinitely, so that the point of contact recedes to infinity, the limiting line approached by (1) must by definition be the asymptote.

Since P lies on the hyperbola, we have

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1;$$

this may be written in the form

$$y_1 = \pm \frac{bx_1}{a}\sqrt{1 - \frac{a^2}{x_1^2}}.$$

Substituting this value of y_1 in (1), we get

$$\frac{x_1 x}{a^2} \mp \frac{x_1 y}{ab}\sqrt{1 - \frac{a^2}{x_1^2}} = 1.$$

If we divide through by x_1 and (for convenience) multiply through by a, this becomes

$$(2) \qquad \frac{x}{a} \mp \frac{y}{b}\sqrt{1 - \frac{a^2}{x_1^2}} = \frac{a}{x_1}.$$

If now x_1 becomes infinite, (2) approaches the form

$$\frac{x}{a} \mp \frac{y}{b} = 0,$$

or

$$y = \pm \frac{b}{a} x.$$

129. Single equation representing the pair of asymptotes. The equations of the asymptotes to the standard hyperbola

$$(1) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

viz.

$$y = \pm \frac{b}{a} x,$$

may be written in the form

$$\frac{x}{a} - \frac{y}{b} = 0, \qquad \frac{x}{a} + \frac{y}{b} = 0.$$

If these equations be multiplied together, we get as the single equation representing the pair of asymptotes (cf. § 21)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \tag{2}$$

This equation differs from that of the hyperbola only in the constant term. It is easily shown that the same result holds in general; that is, that *the equation of the hyperbola in any form differs* * *from the equation of its pair of asymptotes only in the constant term.*

This result affords a valuable check on the answer when the asymptotes of a given hyperbola have been found. It also finds application in a variety of other ways, of which the following is an

Example: Find the equation of the hyperbola having the lines

$$x - y + 1 = 0, \quad x + y + 1 = 0$$

as asymptotes, and passing through (2, 2).

The required equation will have the form

$$(x - y + 1)(x + y + 1) + k = 0.$$

Substituting the coördinates (2, 2), we find

$$1 \cdot 5 + k = 0,$$

or

$$k = -5,$$

whence the result is

$$(x - y + 1)(x + y + 1) - 5 = 0;$$

that is,

$$x^2 - y^2 + 2x - 4 = 0.$$

EXERCISES

1. Using the quadratic discriminant, find the tangents of slope $\pm \frac{b}{a}$ to the standard hyperbola, thus obtaining a new derivation of the equations of the asymptotes.

2. Using the quadratic discriminant, find the tangents to the hyperbola $xy - 3x + 4y + 6 = 0$ (*a*) parallel to Oy; (*b*) parallel to Ox. What lines are these?

---

* More precisely, *may be made to differ.*

3. Prove analytically that if a line is parallel to an asymptote of a hyperbola it intersects the curve in one (finite) point; and conversely.

4. Prove both geometrically and analytically that if two hyperbolas have one asymptote in common and the other asymptotes parallel to each other, they intersect in one and only one finite point. Is the converse true?

5. Investigate the intersection of two hyperbolas if the asymptotes of one are respectively parallel to those of the other.

6. Prove that if an asymptote of one hyperbola is parallel to an asymptote of another hyperbola, the curves intersect in either one or three real and finite points.

7. If two hyperbolas intersect in but two finite points either real or imaginary, what can be said of their asymptotes?

8. Prove that a hyperbola is determined by three points and the directions of the asymptotes.

Find the equations of the following hyperbolas by the method of § 92, using as two of the points the points at infinity on the given lines. (Cf. examples (*a*), (*b*) of § 125.)

9. Through (2, 0), (− 2, 0), (1, 1), with asymptotes parallel to the lines $x \pm 2y = 0$. *Ans.* $x^2 - 4y^2 + 7y = 4$.

10. Through (0, 0), (1, − 1), (− 1, 2), with asymptotes parallel to the lines $x = 0$, $x + 3y = 5$. *Ans.* $x^2 + 3xy + 9x + 7y = 0$.

11. Through (0, 0), (2, 1), (− 3, 2), with asymptotes parallel to the coördinate axes. *Ans.* $7xy - 10x + 6y = 0$.

12. Through (2, 0), (3, 2), (1, − 3), with asymptotes parallel to the lines $2x + y = 6$, $x - 3y = 4$.
Ans. $2x^2 - 5xy - 3y^2 + 132x - 50y - 272 = 0$.

13. A point moves so that the product of its distances from two intersecting lines is constant. Prove that its locus is a hyperbola. What relation do the given lines bear to the curve?

14. A point moves so that the product of its distances from the lines $y = \pm x$ is equal to 2. Find the equation of its locus.

15. A point moves so that the product of its distances from the lines $x + 3y = 6$, $x + y = 5$ is equal to $4\sqrt{5}$. Find the equation of its locus.

16. Prove that the product of the distances of any point of a hyperbola from the asymptotes is constant. (Cf. Ex. 13.)

Find the equations of the following hyperbolas.

17. Through (3, 1), having as asymptotes the lines $x = \pm 2y$.

18. Through $(\frac{1}{3}, -\frac{3}{2})$, having the asymptotes $3x - 2y + 4 = 0$, $y = 0$.

19. Through (0, 0), having the asymptotes $2x - 4y + 7 = 0$, $x + 2y = 2$.

20. Through $(-\frac{1}{2}, \frac{1}{2})$, having the asymptotes $3x - y + 6 = 0$, $x + 4y = 6$.

21. Touching the line $y = 2x - 5$, having the asymptotes $x + y = 0$, $3x - y = 5$. *Ans.* $36x^2 + 24xy - 12y^2 - 60x - 60y + 25 = 0$.

22. Touching the x-axis, having the asymptotes $x - 3y = 1$, $x + y = 0$. Find the point of contact.

Ans. $4x^2 - 8xy - 12y^2 - 4x - 4y + 1 = 0$.

130. A general method for determination of asymptotes. The method used in § 128 for determining asymptotes is difficult of application in most cases, since it requires us to know the general equation of the tangent in advance. A method that is simple and at the same time very general, since it applies to curves of any degree, will first be illustrated by an

Example: Find the asymptotes of the hyperbola

$$y^2 - xy + 2x - 2y + 3 = 0. \tag{1}$$

Let us assume the equation of the asymptote in the form

$$y = mx + k, \tag{2}$$

and try to determine m and k so that this line shall intersect the curve in two points at infinity. Substituting the value of y from (2) in (1) and collecting terms, we get

$$(m^2 - m)x^2 + (2km - k + 2 - 2m)x + k^2 - 2k + 3 = 0.$$

The roots of this quadratic in x are the abscissas of the points of intersection of the line and the curve: these roots will be infinite, and the two points of intersection will lie at infinity, if the coefficients of x^2 and x vanish — i.e. if

$$m^2 - m = 0, \tag{3}$$

$$2km - k + 2 - 2m = 0. \tag{4}$$

By (3),

$$m = 0 \text{ or } 1;$$

by (4), the corresponding values of k are

$$k = 2 \text{ or } 0.$$

Thus the asymptotes are

$$y = 2,$$

$$y = x.$$

Check: The single equation representing the asymptotes is

$$(y - 2)(y - x) = 0,$$

or

$$y^2 - xy + 2x - 2y = 0,$$

which differs from the equation of the hyperbola only in the constant.

The method may now be summarized as follows:

To find the asymptotes of a hyperbola, substitute

$$y = mx + k$$

in the equation of the curve, equate to 0 *the coefficients of* x^2 *and* x*, and solve for* m *and* k.

This method is open to some objection, in that it does not rest squarely upon the definition of asymptote. Instead of finding the limiting position of the tangent, as was done in § 128, it determines a line intersecting the curve in two points at infinity, which is not necessarily the same thing. However, it is easily shown that for the hyperbola the rule always gives correct results (see Ex. 14 below).

The method just described evidently fails to detect asymptotes parallel to Oy. But if a hyperbola has an asymptote parallel to Oy, its equation cannot contain the term in y^2 — i.e. it must have the form

$$Ax^2 + Bxy + Dx + Ey + F = 0.$$

This equation when solved for y becomes

$$y = -\frac{Ax^2 + Dx + F}{Bx + E},$$

which shows that the line

$$Bx + E = 0$$

is the asymptote parallel to Oy.

EXERCISES

Find the asymptotes of the following hyperbolas; check the results.

1. $x^2 - y^2 + 4x - 4y + 6 = 0$.

2. $3x^2 - y^2 + 6x - 2y + 3 = 0$.

3. $2xy - y^2 - 2x + 4y = 0$. Trace the curve.

4. $6x^2 - 7xy - 3y^2 = 3$. Trace the curve by composition of ordinates.

5. $15x^2 + 16xy + 4y^2 - 4x - 4y = 1$.

6. $4x^2 - 2xy - 2y^2 + 4x + 5y + 5 = 0$.

7. $11x^2 + 7xy - 5y^2 + 5 = 0$. *Ans.* $y = (\frac{7}{10} \pm \frac{1}{10}\sqrt{269})\,x$.

8. $x^2 - 3xy + 6x + 5 = 0$. Trace the curve.

9. $6xy - 15x - 8y + 10 = 0$.

10. $3x^2 - 5xy + 8x - 9y + 6 = 0$.

11. $21x^2 - 11xy - 40y^2 - 26x - 55y + 1 = 0$.

12. $xy + 8x - 5y + 6 = 0$.

13. Prove that if the term in y^2 is missing from its equation, a hyperbola has an asymptote parallel to the y-axis, and conversely.

14. Find the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by the method of § 130.

15. Prove that if the equation of a hyperbola contains no terms of the first degree, the asymptotes can be found by equating to 0 the group of terms of the second degree and factoring the resulting equation.

131. Equilateral hyperbola referred to its asymptotes. Since the asymptotes of the equilateral, or rectangular, hyperbola are perpendicular to each other (§ 81), they may be taken as the axes of coördinates. In that case, the asymptotes are the lines

$$x = 0, \quad y = 0;$$

it follows by the theorem of § 129 that *the equation*

$$(10) \qquad xy = k$$

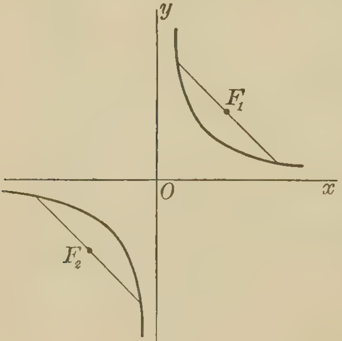


Fig. 105

represents a rectangular hyperbola whose asymptotes are the coordinate axes.

If k is positive, the curve lies in the first and third quadrants; its equation may be written in the more convenient form

$$xy = c^2. \tag{11}$$

The axes of this curve lie in the lines $y = x$, $y = -x$; the vertices are the points (c, c), $(-c, -c)$; the semi-axis is

$$a = \sqrt{2}\, c,$$

and the foci are at the distance $2c$ from the center.

EXERCISES

1. Prove that the hyperbola $xy = k$ lies in the first and third or the second and fourth quadrants according as k is positive or negative.

2. Trace the hyperbola $xy = 4$.

3. Trace the hyperbola $xy + 6 = 0$.

4. Find the tangent and normal to the hyperbola $xy + 2 = 0$ at the point $x = 1$.

5. Find the tangent and normal to the hyperbola $2xy = 3$ at the point $(-3, -\frac{1}{2})$.

6. Find the tangents to the hyperbola $xy = -3$ parallel to the line $x - 2y + 3 = 0$.

7. Find the tangents to the hyperbola $xy = 6$ perpendicular to the line $3x - 2y = 5$.

8. Find the tangents to the hyperbola $xy = 4$ through the point $(1, -2)$.

9. Find the tangents to the hyperbola $5xy + 8 = 0$ through the point $(0, 3)$.

10. Find the tangent to the hyperbola $xy = c^2$ at any point (x_1, y_1) on the curve. *Ans.* $y_1x + x_1y = 2c^2$.

11. Prove that the equations $x = c \cot \phi$, $y = c \tan \phi$ represent a rectangular hyperbola referred to its asymptotes, and construct the angle ϕ corresponding to any point of the curve.

12. What is represented by the equations $x = c \sec \phi$, $y = c \csc \phi$? Can the angle ϕ always be constructed?

EXERCISES

Find the asymptotes of the following hyperbolas; check the results.

1. $x^2 - y^2 + 4x - 4y + 6 = 0.$

2. $3x^2 - y^2 + 6x - 2y + 3 = 0.$

3. $2xy - y^2 - 2x + 4y = 0.$ Trace the curve.

4. $6x^2 - 7xy - 3y^2 = 3.$ Trace the curve by composition of ordinates.

5. $15x^2 + 16xy + 4y^2 - 4x - 4y = 1.$

6. $4x^2 - 2xy - 2y^2 + 4x + 5y + 5 = 0.$

7. $11x^2 + 7xy - 5y^2 + 5 = 0.$ *Ans.* $y = (\frac{7}{10} \pm \frac{1}{10}\sqrt{269})\,x.$

8. $x^2 - 3xy + 6x + 5 = 0.$ Trace the curve.

9. $6xy - 15x - 8y + 10 = 0.$

10. $3x^2 - 5xy + 8x - 9y + 6 = 0.$

11. $21x^2 - 11xy - 40y^2 - 26x - 55y + 1 = 0.$

12. $xy + 8x - 5y + 6 = 0.$

13. Prove that if the term in y^2 is missing from its equation, a hyperbola has an asymptote parallel to the y-axis, and conversely.

14. Find the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by the method of § 130.

15. Prove that if the equation of a hyperbola contains no terms of the first degree, the asymptotes can be found by equating to 0 the group of terms of the second degree and factoring the resulting equation.

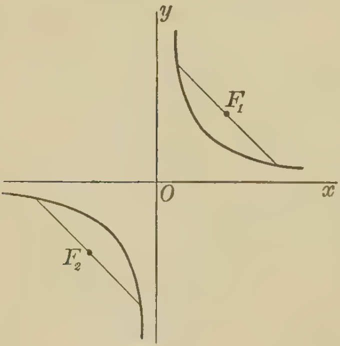


Fig. 105

131. Equilateral hyperbola referred to its asymptotes. Since the asymptotes of the equilateral, or rectangular, hyperbola are perpendicular to each other (§ 81), they may be taken as the axes of coördinates. In that case, the asymptotes are the lines

$$x = 0, \quad y = 0;$$

it follows by the theorem of § 129 that *the equation*

$$xy = k \qquad (10)$$

represents a rectangular hyperbola whose asymptotes are the coördinate axes.

If k is positive, the curve lies in the first and third quadrants; its equation may be written in the more convenient form

$$xy = c^2. \tag{11}$$

The axes of this curve lie in the lines $y = x$, $y = -x$; the vertices are the points (c, c), $(-c, -c)$, the semi-axis is

$$a = \sqrt{2}\,c,$$

and the foci are at the distance $2c$ from the center.

EXERCISES

1. Prove that the hyperbola $xy = k$ lies in the first and third or the second and fourth quadrants according as k is positive or negative.

2. Trace the hyperbola $xy = 4$.

3. Trace the hyperbola $xy + 6 = 0$.

4. Find the tangent and normal to the hyperbola $xy + 2 = 0$ at the point $x = 1$.

5. Find the tangent and normal to the hyperbola $2xy = 3$ at the point $(-3, -\frac{1}{2})$.

6. Find the tangents to the hyperbola $xy = -3$ parallel to the line $x - 2y + 3 = 0$.

7. Find the tangents to the hyperbola $xy = 6$ perpendicular to the line $3x - 2y = 5$.

8. Find the tangents to the hyperbola $xy = 4$ through the point $(1, -2)$.

9. Find the tangents to the hyperbola $5xy + 6 = 0$ through the point $(0, 3)$.

10. Find the tangent to the hyperbola $xy = c^2$ at any point (x_1, y_1) on the curve. *Ans.* $y_1x + x_1y = 2c^2$.

11. Prove that the equations $x = c \cot \phi$, $y = c \tan \phi$ represent a rectangular hyperbola referred to its asymptotes, and construct the angle ϕ corresponding to any point of the curve.

12. What is represented by the equations $x = c \sec \phi$, $y = c \csc \phi$? Can the angle ϕ always be constructed?

A line parallel to a coördinate axis is considered to be directed, the positive sense agreeing with that of the parallel axis; unless the contrary is indicated, lines oblique to the coördinate axes are undirected.

134. Distance between two points. To find the distance d between two points P_1: (x_1, y_1, z_1) and P_2: (x_2, y_2, z_2), we note that, in Fig. 108,

$$d = \sqrt{\overline{P_1M}^2 + \overline{MP_2}^2}.$$

But

$$\overline{P_1M}^2 = \overline{P_1L}^2 + \overline{LM}^2,$$

so that

$$d = \sqrt{\overline{P_1L}^2 + \overline{LM}^2 + \overline{MP_2}^2}.$$

Now

$$P_1L = IJ = GH = x_2 - x_1;$$

similarly,

$$LM = y_2 - y_1,$$
$$MP_2 = z_2 - z_1.$$

Thus

$$(1) \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

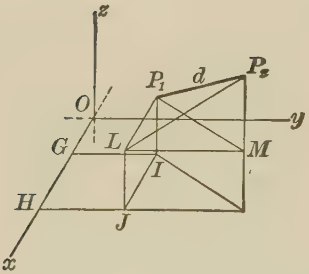


FIG. 108

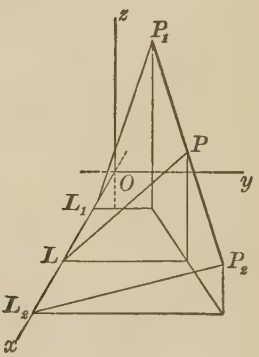


FIG. 109

135. Midpoint of a line segment. To find the point P: (x, y, z) bisecting the line segment joining the points P_1: (x_1, y_1, z_1) and P_2: (x_2, y_2, z_2), let us drop perpendiculars P_1L_1, PL, P_2L_2 from these points to Ox. Then (Fig. 109)

$$OL = OL_1 + \tfrac{1}{2} L_1L_2;$$

that is,

$$x = x_1 + \tfrac{1}{2}(x_2 - x_1) = \tfrac{1}{2}(x_1 + x_2).$$

Proceeding similarly for y and z, we obtain the following formulas for the coördinates of the midpoint:

$$\begin{cases} x = \tfrac{1}{2}(x_1 + x_2), \\ y = \tfrac{1}{2}(y_1 + y_2), \\ z = \tfrac{1}{2}(z_1 + z_2). \end{cases} \tag{2}$$

EXERCISES

1. Plot the points (3, 4, 1), (1, 3, 5), (3, 1, 6), (1, 6, 1).

2. Plot the points (– 2, 3, 2), (– 2, – 1, 3), (2, 4, – 3), (– 2, – 3, – 6).

3. Plot the points (0, 0, – 5), (– 4, 0, – 3), (– 2, 2, 0), (0, – 2, 0).

4. Through the point (1, 3, 5) draw lines intersecting each of the coördinate axes at right angles.

5. Draw perpendiculars from the point (– 2, 3, 1) to each of the coördinate planes.

6. From the point (3, – 2, 6) draw perpendiculars (*a*) to each coördinate axis, (*b*) to each coördinate plane.

7. Given any point in the first octant, with the foot of the perpendicular from this point to the y-axis, draw perpendiculars from the point to each coördinate axis and to each coördinate plane.

8. In each coördinate plane, draw a line through O making the angle arctan 2 with one of the axes, and a perpendicular to this line through an arbitrary point in that plane.

9. In each coördinate plane, draw a circle of radius a with center at O.

10. In the xy-plane, plot the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

11. In the zx-plane, plot the parabola $z^2 = x$.

12. Draw a rectangular parallelepiped, or *box*, with its edges parallel to the coördinate axes and having the points (0, 0, 0), (3, 4, 6) as ends of a diagonal.

13. Draw a box with its edges parallel to the coördinate axes and having the points (1, 3, 2), (3, 6, 7) as ends of a diagonal.

14. Where is a point situated if

(*a*) $x = 0$? (*b*) $z = 0$? (*c*) $x = y = 0$?
(*d*) $y = z = 0$? (*e*) $x = 2$? (*f*) $x = 2, y = 1$?
(*g*) $x = z$? (*h*) $y = z, x = 0$? (*i*) $x = y = z$?

Find the distances between the following pairs of points. Draw the figures.

15. $(1, 3, -2), (3, 5, 3)$.

16. $(0, \frac{1}{2}, -2), (\frac{3}{2}, -1, -\frac{5}{2})$.

17. $(-5, -3, -2), (0, -6, -5)$.

18. $(13, -8, 16), (-12, -18, -4)$.

19. Prove that the points $(1, 3, 3)$, $(0, 1, 0)$, $(4, -1, 0)$ form a right triangle, and find its area.

20. Prove that the points $(2, -15, -4)$, $(-3, -5, -9)$, $(2, 5, 6)$ form a right triangle, and find its area.

21. Prove that the triangle whose vertices are $(3, 1, 2)$, $(1, 0, -1)$, $(2, -1, 5)$ is isosceles, and find its area. *Ans.* $\frac{3}{2}\sqrt{19}$.

22. Prove that the points $(6, 1, -3)$, $(0, -2, 3)$, $(10, 3, -7)$ lie in a straight line. Check by plotting.

23. Do the points $(0, \frac{1}{3}, 1)$, $(1, 0, 0)$, $(2, -\frac{1}{3}, -1)$ lie in a straight line?

24. Do the points $(13, 10, -7)$, $(-3, -10, 11)$, $(1, -5, 7)$ lie in a straight line?

25. Prove that the sphere with center at $(1, 3, 2)$ passing through $(6, -2, 3)$ also passes through $(0, 2, -5)$.

Find the midpoints of the segments joining the following pairs of points.

26. $(1, 4, 4), (2, -3, -5)$.

27. $(\frac{5}{2}, -\frac{3}{2}, -\frac{5}{4}), (3, \frac{7}{4}, -\frac{3}{4})$

28. $(-\frac{1}{4}, \frac{3}{5}, -\frac{2}{3}), (\frac{5}{3}, \frac{1}{2}, -\frac{2}{5})$.

29. $(0, 3.7, -1.4), (2.15, -1, 3.2)$.

30. Find the lengths of the medians of the triangle whose vertices are $(1, 2, 3)$, $(3, -4, 1)$, $(-5, 0, -1)$. *Ans.* $\sqrt{59}, \sqrt{29}, 5\sqrt{2}$.

31. Solve Ex. 19 in another way.

32. Solve Ex. 20 in another way.

33. The line from $(2, -1, 3)$ to $(5, 4, 2)$ is doubled. Find the endpoint. Draw the figure. *Ans.* $(8, 9, 1)$.

34. The line from $(0, -1, 6)$ to $(1, 3, 1)$ is trebled. Find the endpoint.

35. The line from $(3, 3, 2)$ to $(0, 0, -2)$ is quadrupled. Find the endpoint.

36. Prove that the points $(2, 1, 4)$, $(0, 0, 0)$, $(3, -1, 2)$, $(5, 0, 6)$ form a parallelogram.

37. Do the points $(-1, 2, 3)$, $(2, 4, 3)$, $(-1, 6, 3)$, $(-1, 4, 6)$ form a parallelogram? *Ans.* No.

38. Prove that the points $(3, 2, 2)$, $(1, 2, 1)$, $(2, 4, -1)$, $(4, 4, 0)$ form a rectangle. Draw the figure.

39. What is the distance of the point (x, y, z) from Ox? From Oy? From Oz? From O?

40. A point is at the distance 3 from the zx-plane, 5 from the x-axis, and $\sqrt{26}$ from the origin. Find its coördinates.

41. A point is at the distance 2 from Ox, $\sqrt{5}$ from Oy, and $\sqrt{7}$ from Oz. Find its coördinates.

42. Prove that the lines joining the midpoints of the opposite sides of any quadrilateral (not necessarily plane) intersect and bisect each other.

136. Direction angles; direction cosines.

Given any directed line λ passing through the origin, the angles α, β, γ formed by this line with the x-, y-, and z-axes are called the *direction angles* of the line, and the cosines of these angles are the *direction cosines*. Direction cosines are denoted by the letters l, m, n respectively:

$$l = \cos\alpha, \; m = \cos\beta, \; n = \cos\gamma.$$

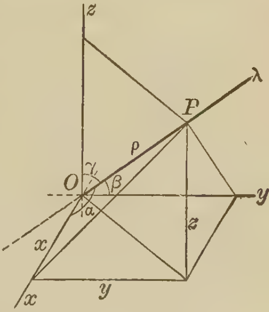


FIG. 110

More generally, if the given line does not pass through the origin, its direction angles and direction cosines are defined as *equal to those of the parallel line through the origin.*

Given the direction cosines l, m, n of any directed line λ', let $P: (x, y, z)$ be any point on the parallel λ through the origin, and denote the distance OP by ρ. Evidently (Fig. 110)

$$x = OP \cos \alpha = l\rho;$$

similarly,

$$y = m\rho, \ z = n\rho.$$

Thus the coördinates of P can be found if l, m, n are known, so that the line λ is determined. It follows that *the direction of any line is determined if its direction cosines are given.*

If the positive sense on the line be reversed, its direction angles are replaced by their supplements, and the signs of the direction cosines are changed. Hence the direction cosines of an undirected line are ambiguous in sign: if for one sense on the line they are l, m, n, for the opposite sense they are $-l, -m, -n$.

In Fig. 110, we have

$$x^2 + y^2 + z^2 = \rho^2,$$

or

$$l^2\rho^2 + m^2\rho^2 + n^2\rho^2 = \rho^2.$$

Hence the important

THEOREM: *The direction cosines of any line are connected by the relation*

$$\mathbf{l^2 + m^2 + n^2 = 1.} \tag{3}$$

137. Direction cosines proportional to three numbers. If the direction cosines of a line are proportional to three numbers a, b, c, their actual values must be

$$l = \frac{a}{s}, \quad m = \frac{b}{s}, \quad n = \frac{c}{s},$$

where s is a quantity as yet undetermined. Substituting these values of l, m, and n in (**3**), § 136, we get

$$\frac{a^2}{s^2} + \frac{b^2}{s^2} + \frac{c^2}{s^2} = 1,$$

or

$$a^2 + b^2 + c^2 = s^2,$$

so that

$$s = \sqrt{a^2 + b^2 + c^2}.$$

Hence:

If the direction cosines of a line are proportional to any three numbers a, b, c, their actual values are

$$(4)\quad l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

The ambiguity of sign mentioned in § 136 appears here in the fact that we might equally well choose the negative sign before the radical.

138. Radius vector of a point. The directed segment OP from the origin to the point $P: (x, y, z)$ is called the *radius vector* of P; its length is

$$(5)\qquad \rho = \sqrt{x^2 + y^2 + z^2}.$$

By § 136, the coördinates of the point in terms of the radius vector and its direction cosines are

$$(6)\qquad x = l\rho,\ y = m\rho,\ z = n\rho.$$

Hence *the direction cosines of the radius vector of a point are proportional to the coördinates of the point.*

139. Direction cosines of the line through two points. Let d be the distance between the points $P_1: (x_1, y_1, z_1)$ and $P_2: (x_2, y_2, z_2)$. Then the direction cosines of the line P_1P_2 are

$$l = \frac{P_1L}{d} = \frac{x_2 - x_1}{d},$$

$$m = \frac{P_1M}{d} = \frac{y_2 - y_1}{d},$$

$$n = \frac{P_1N}{d} = \frac{z_2 - z_1}{d}.$$

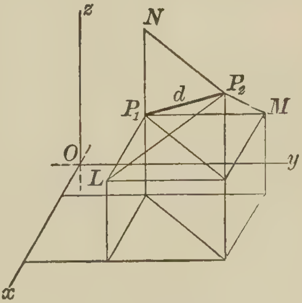


FIG. 111

Hence:

The direction cosines of the line joining the points $P_1: (x_1, y_1, z_1)$ and $P_2: (x_2, y_2, z_2)$ are proportional to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

EXERCISES

1. A line makes an angle of 45° with Ox and 60° with Oz. What angle does it make with Oy?

2. A line makes the angles $\arccos \frac{1}{3}$, $\arccos \frac{1}{2}$ with Ox and Oy respectively. What angle does it make with Oz?

3. A line makes angles of 45° with Oy and Oz. What angle does it make with Ox?

4. A line makes equal angles with the axes. Find its direction cosines.

Draw the following lines.

5. Through O making the angle 45° with Ox and 60° with Oz. (See Ex. 1.) How many such lines are there?

6. Through O, having $l = \frac{1}{9}$, $m = \frac{4}{9}$.

7. Through O, having $m = -\frac{2}{3}$, $n = \frac{2}{3}$.

8. Through (3, 4, 2), having $l = \frac{1}{2}\sqrt{2}$, $m = \frac{1}{2}\sqrt{2}$.

9. Through (4, 2, 5), having $l = \frac{3}{4}$, $n = \frac{1}{2}$.

10. Through (2, 5, 3), having $m = -\frac{1}{3}$, $n = -\frac{2}{3}$.

11. Can a line be drawn making an angle of 30° with Ox and 45° with Oz?

12. Prove that it is impossible to draw a line two of whose direction angles shall be less than 45°.

13. The direction cosines of a line are proportional to $1, -3, 2$. Find their values.

14. The direction cosines of a line are proportional to $-2, 2, -5$. Find their values.

15. For each of the following points, find the length and direction cosines of the radius vector: (*a*) (2, 3, 3); (*b*) $(-3, 5, -2)$; (*c*) (2, 0, 0); (*d*) $(\frac{4}{9}, \frac{1}{9}, \frac{8}{9})$.

16. Where must a point lie if its radius vector has (*a*) $l = 0$? (*b*) $l = m = 0$? (*c*) $l = \frac{1}{2}$? (*d*) $l = 1$? (*e*) $m = n = \frac{1}{2}\sqrt{2}$?

17. A point is at the distance 3 from O, and its radius vector makes an angle of 60° with Oy and 45° with Oz. Find its coördinates in two ways. *Ans.* $(\pm \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\sqrt{2})$.

18. A line makes an angle of 60° with Ox and 60° with Oz. What angle does it make with Oy?

19. A point is at the distance 4 from the yz-plane, and its radius vector has the direction angles $\beta = \frac{\pi}{6}$, $\gamma = \frac{\pi}{2}$. Find its coördinates, and draw the figure. *Ans.* $(4, 4\sqrt{3}, 0)$.

20. A point is at the distance 3 from O, and the direction cosines of its radius vector are proportional to 1, 2, 4. Find its coördinates.

21. A point lies on a sphere of radius 5 described about the origin. If it is at a distance 4 from the xy-plane and its radius vector makes an angle of $\frac{\pi}{3}$ with Ox, find its coördinates. *Ans.* $(\frac{5}{2}, \pm \frac{1}{2}\sqrt{11}, 4)$.

22. A point is at the distance 2 from the plane of yz, and its radius vector has $\alpha = 60°$, $\beta = 45°$. Find its coördinates.

23. A point is at the distance 4 from O, at the distance -1 from the zx-plane, and its radius vector makes the angle $\frac{\pi}{4}$ with Ox. Find its coördinates. *Ans.* $(2\sqrt{2}, -1, \pm\sqrt{7})$.

24. A line has direction angles so related that $\alpha = \beta = \frac{1}{2}\gamma$. Find α, β, γ. *Ans.* $\alpha = \beta = \frac{\pi}{4}$, $\gamma = \frac{\pi}{2}$, or $\alpha = \beta = \frac{\pi}{2}$, $\gamma = \pi$.

25. A point is at the distance $2\sqrt{5}$ from Oy, 8 from O, and its radius vector makes an angle $\frac{\pi}{3}$ with Ox. Find its coördinates.

Ans. $(4, \pm 2\sqrt{11}, \pm 2)$.

26. A point is at the distance $\sqrt{5}$ from the z-axis, 1 from the xy-plane, and its radius vector makes an angle $\frac{\pi}{4}$ with Ox. Find its coordinates.

27. A point is at the distance $\sqrt{2}$ from Oz, $\sqrt{3}$ from Ox, and its radius vector makes an angle of 45° with Oz. Find its coördinates.

28. Find the direction cosines of the line (a) joining the points $(3, -1, 2)$, $(4, 0, -3)$; (b) joining the points $(-1, -\frac{5}{2}, 6)$, $(-1, \frac{3}{4}, \frac{5}{8})$.

29. Find the direction cosines of the sides of the triangle whose vertices are $(1, -3, 3)$, $(5, 1, -6)$, $(-4, 1, -2)$.

30. Prove in a new way that the points $(6, 1, -3)$, $(0, -2, 3)$, $(10, 3, -7)$ lie in a straight line. (Ex. 22, p. 212.)

31. Determine in a new way whether the points $(0, \frac{1}{3}, 1)$, $(1, 0, 0)$, $(2, -\frac{1}{3}, -1)$ lie in a line. (Ex. 23, p. 212.)

32. Do the points $(2, 4, -3)$, $(3, -2, 4)$, $(5, -14, 16)$ lie in a line?

33. Prove in another way that the points $(2, 1, 4)$, $(0, 0, 0)$, $(3, -1, 2)$, $(5, 0, 6)$ form a parallelogram. (Ex. 36, p. 213.)

34. Prove in another way that the points $(3, 2, 2)$, $(1, 2, 1)$, $(2, 4, -1)$, $(4, 4, 0)$ form a rectangle. (Ex. 38, p. 213.)

140. Projections. The *projection* of a point P upon any line is the foot of the perpendicular from P to that line. The projection of a line segment P_1P_2 upon any line is the segment joining the projections of the end points P_1, P_2 upon that line. In Fig. 109, p. 210, L_1L_2 is the projection of the segment P_1P_2 upon the x-axis.

The projection of a broken line upon any line is the sum of the projections of the segments forming the broken line. *The projection of a broken line $P_1P_2 \dots P_n$ upon any line is equal to the projection of the closing line P_1P_n upon that line.* For instance, in Fig. 112,

$$L_1L_2 + L_2L_3 + L_3L_4 = L_1L_4.$$

It should be noted that the segments composing the broken line need not all lie in the same plane.

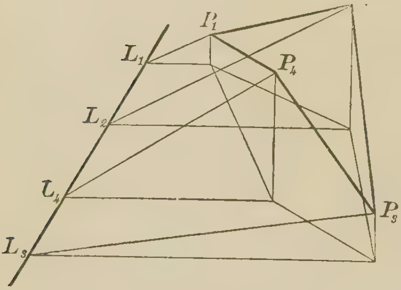


FIG. 112

141. Angle between two lines. To find the angle α between any two intersecting lines λ_1, λ_2, let us take the point of intersection as the origin. (This involves no loss of generality, since if the lines intersect at any other point we can find the angle between parallels to the given lines through the origin.) Denote the direction cosines of λ_1 by l_1, m_1, n_1, those of λ_2 by l_2, m_2, n_2, and choose on λ_1 any point $P: (x, y, z)$ having the radius vector ρ. Then, in Fig. 113,

$$OL = x = l_1\rho, \quad LM = y = m_1\rho, \quad MP = z = n_1\rho.$$

Now the projection on λ_2 of the broken line $OLMP$ equals the projection on λ_2 of the closing line OP. Hence

$$OP \cos\phi = OL \cdot l_2 + LM \cdot m_2 + MP \cdot n_2,$$

or

$$\rho \cos\phi = l_1\rho l_2 + m_1\rho m_2 + n_1\rho n_2:$$

that is,

$$\cos\phi = l_1 l_2 + m_1 m_2 + n_1 n_2. \tag{7}$$

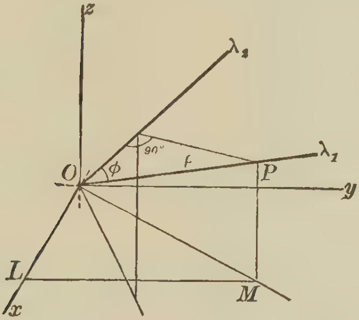


Fig. 113

It is useful to extend the idea of angle between two lines to the case of non-intersecting lines. The angle between two non-intersecting lines is defined as being *equal to the angle between two intersecting lines that are respectively parallel to the given lines.* With this convention formula (7) gives the angle between any two lines in space.

142. Perpendicular lines. In formula (7), if $\cos\phi = 0$, then $\phi = 90°$, and we have the

THEOREM: *Two lines having the direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular if*

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \tag{8}$$

COROLLARY: *Two lines whose direction cosines are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular if*

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0. \tag{9}$$

EXERCISES

1. Find the projections upon the axes of the points (*a*) $(3, 5, 2)$, (*b*) $(4, -1, -5)$, (*c*) $(2, 0, -3)$, (*d*) $(3, 0, 0)$.

2. Find the lengths of the projections upon the axes of the line from $(2, 4, 3)$ to $(5, 1, 2)$. *Ans.* $3, -3, -1$.

3. Find the lengths of the projections upon the axes of the line from (x_1, y_1, z_1) to (x_2, y_2, z_2).

4. Find the projections upon the axes of the broken line joining the points $(1, 2, 1)$, $(3, 5, 5)$, $(4, 1, 3)$, $(6, 6, 8)$, and of the closing line. Draw the figure.

5. Prove that the length of the projection of a closed polygon upon any line is 0.

6. Prove that if two broken lines have the same initial and terminal points, their projections upon any line are equal.

7. Find the angle between two lines whose direction cosines are proportional to $2, 1, -2$ and $1, 1, 0$. *Ans.* 45°.

8. Find the angle between two lines whose direction cosines are proportional to $1, 2, 3$ and $2, 3, 6$. *Ans.* $\text{Arccos}\,\frac{13}{49}\sqrt{14}$.

9. Find the angle between the radius vectors of the points $(3, -4, 5)$ and $(-1, -1, 0)$. *Ans.* $\text{Arccos}\,\frac{1}{10}$.

10. Find the angle between the radius vectors of the points $(2, 5, -7)$ and $(6, -5, -2)$. *Ans.* $\text{Arccos}\,\dfrac{1}{13\sqrt{30}}$.

11. Find the angle between the radius vectors of the points $(1, 2, 3)$ and $(-5, 4, -1)$. *Ans.* $\dfrac{\pi}{2}$.

12. Find the angle between the lines joining the points $(2, -3, 4)$, $(4, 2, 0)$ and $(5, 3, -1)$, $(-2, 2, 2)$.

13. Find the angles of the triangle whose vertices are $(3, 1, 3)$, $(5, -2, -6)$, $(0, 0, 3)$.

14. Prove that if equal segments of length ρ be laid off from O on the two lines of § 141, the ends of these segments are the points P_1: $(l_1\rho, m_1\rho, n_1\rho)$, P_2: $(l_2\rho, m_2\rho, n_2\rho)$. Hence derive formula (**7**) by applying the cosine law of trigonometry to the triangle OP_1P_2.

15. Prove the corollary of § 142.

16. Prove in another way that the points $(1, 3, 3)$, $(0, 1, 0)$, $(4, -1, 0)$ form a right triangle. (Ex. 19, p. 212.)

17. Prove in two new ways that the points $(3, 1, 2)$, $(1, 0, -1)$, $(2, -1, 5)$ form an isosceles triangle. (Ex. 21, p. 212.)

18. Prove in several ways that the points $(3, 4, 4)$, $(-1, 3, 5)$, $(-1, 6, 8)$, $(3, 7, 7)$ are the vertices of a square.

143. Cylindrical coördinates. Given a point P in space, let us drop a perpendicular from P to a point M in the xy-plane. The position of P is evidently determined if we know the polar coördinates r, θ of M together with the z-coördinate MP. This combination of polar coördinates in the xy-plane with the cartesian z is the so-called *cylindrical* system. The coördinates are written in parenthesis in the order (r, θ, z).

Evidently the only formulas required for transformation from cylindrical to cartesian coördinates, or vice versa, are those of § 65.

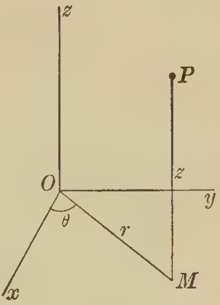


FIG. 114

EXERCISES

1. Find the cylindrical coördinates of the points (a) $(1, \sqrt{3}, 1)$, (b) $(2, 2, 5)$, (c) $(3, 0, -3)$.

2. Plot the points whose cylindrical coördinates are (a) $(1, \frac{\pi}{3}, 2)$, (b) $(3, \frac{3}{4}\pi, 3)$, (c) $(4, \pi, -1)$.

3. Find the cartesian coördinates of the points of Ex. 2.

4. What is the locus of points for which $r = 1$?

5. What is the locus of points for which $\theta = \frac{\pi}{4}$?

CHAPTER XI

LOCI

144. The locus of an equation. If x, y, and z are connected by an equation of any form, we may assign values at pleasure to two of the variables and compute the third, thus determining certain sets of values of x, y, and z satisfying the equation. Each of these sets of values may be considered as the cartesian coördinates of a point. The points whose coördinates satisfy the equation are not scattered at random throughout space; instead, they form in the aggregate a definite *surface*, called the *locus* of the equation. That is:

In space of three dimensions, the locus of an equation is a surface containing those points, and only those points, whose coördinates satisfy the equation.

Of course this definition applies just as well in cylindrical coördinates.

It follows from the above that, in solid geometry, *every equation represents a surface.* If this statement is to hold without exception, however, the idea of surface must be extended beyond its ordinary meaning. Thus in limiting cases a "surface" may reduce to a curve, to one or more separate points, or to an imaginary locus. (Compare the similar extension of "curve" in § 15.) These cases are usually of trivial importance.

In the remainder of our work, attention will be confined to cartesian coördinates except where the contrary is indicated.

Examples: (*a*) If the coördinates of a point satisfy the equation

(1) $$x = y,$$

the point must be equidistant from the yz- and zx-planes. Hence the locus of (1) is a plane through the z-axis bisecting the angle between the yz- and zx-planes.

(*b*) If the coördinates of a point satisfy the equation

$$x^2 + y^2 + z^2 = 25, \tag{2}$$

the point must be at the distance 5 from the origin. Hence the locus of equation (2) is a sphere of radius 5 with center at O.

145. Determination of points on a surface; intercepts on the axes. The method of plotting by separate points, used in plane geometry to trace a curve, is evidently almost useless as a means of presenting a surface to the eye. Nevertheless it is sometimes desirable to determine points on a surface; as suggested in § 144, this is done by *assigning values to two of the variables and computing the third.*

In particular, the *intercepts on the axes* are evidently found by *equating two of the variables to* 0, *and solving for the third:* to find the x-intercepts, set y and z equal to 0, etc.

146. Planes parallel to a coördinate plane. The equation

$$x = 2$$

evidently represents a plane parallel to the yz-plane at a distance $+ 2$ from it; for that plane contains all those points, and only those points, whose x-coördinate is 2. Similarly, any equation of the form

$$x = k$$

represents a plane parallel to the yz-plane. An analogous result holds for an equation of the same form in y or in z. Thus we have the

THEOREM: *An equation of the first degree in one variable represents a plane parallel to the plane of the other two variables; and conversely.*

147. Plane sections of a surface; traces. A plane and a surface intersect in general in a curve, called the *section* of the surface by the plane. Of particular importance are the sections of a surface by the coördinate planes; these sections will be called for brevity the *traces* of the surface.

If in the equation of a surface we set $z = 0$, the resulting equation in x and y, when considered as the equation of a locus in the xy-plane, evidently represents the xy-trace of the surface. Similarly we find the yz-trace by setting $x = 0$, and the zx-trace by setting $y = 0$.

Further, if in the equation of the surface we substitute $z = k$, the resulting equation in x and y, when considered as the equation of a curve in the plane $z = k$, represents the section of the surface by that plane. In the same way the section by any plane parallel to a coördinate plane may be found.

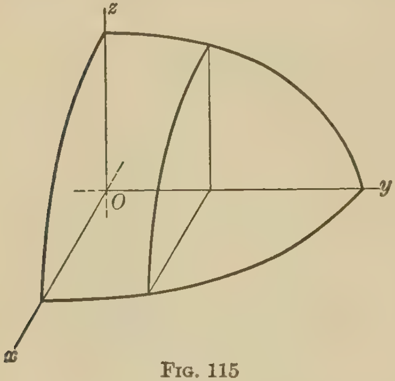


Fig. 115

Example: The surface

$$\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{9} = 1$$

intersects the xy-plane in the ellipse

$$\frac{x^2}{16} + \frac{y^2}{25} = 1,$$

the yz-plane in the ellipse

$$\frac{y^2}{25} + \frac{z^2}{9} = 1,$$

the zx-plane in the ellipse

$$\frac{x^2}{16} + \frac{z^2}{9} = 1,$$

the plane $y = 2$ in the ellipse

$$\frac{x^2}{16} + \frac{z^2}{9} = \frac{21}{25}.$$

Figure 115 shows that part of the surface lying in the first octant. This surface is called an *ellipsoid*; it will be discussed in detail in § 184.

EXERCISES

In the following cases, determine whether the given points lie on the surface.

1. $x^2 + y^2 + z = 0$, $(4, -2, -20)$, $(-1, 0, 1)$.

2. $2x + 4y + z = 8$, $(3, 3, -12)$, $(-1, -1, -2)$

3. $y + 2z - x = 4$, $(4, 6, 1)$, $(-1, 3, 0)$.

4. $y^2 = x$, $(-4, -2, 3)$, $(1, -1, 20)$.

5. $x^2 + y^2 = a^2$, $(-\frac{3}{5}a, -\frac{4}{5}a, \frac{3}{5}a)$, $(0, 0, a)$.

What is represented by each of the following equations?

6. $z^2 = 1$. **7.** $y^3 + 3y^2 + 2y = 0$.

8. $x^2 + y^2 = 0$. **9.** $x^2 + 2x + 2 = 0$.

10. $xyz = 0$. **11.** $yz + 4y - 3z = 12$.

In each of the following cases, find the intercepts on the axes and determine several other points; also draw the traces and one or more other sections.

12. $3x + y + 5z = 6$. **13.** $x + 2y + 3z = 12$.

14. $x - y - 2z = 3$. **15.** $x + z = 6$.

16. $3y - z = 8$. **17.** $2x - y + 4z + 1 = 0$.

18. $x^2 + y^2 + z^2 = 25$. **19.** $y^2 + z^2 = 2x$.

20. $y^2 + z^2 = x^2$. **21.** $y^2 = 4ax$.

22. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **23.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

24. $x^2 + z^2 = a^2$. **25.** $x^2 - y^2 = a^2$.

26. $xy + yz + zx = 0$. **27.** $xy = z^2$.

148. The locus of two simultaneous equations. Two surfaces intersect in general in a curve. This may be a plane curve, but ordinarily its points do not all lie in a plane, in which case it is called a *twisted*, or *skew*, curve.

Given the equations of two surfaces, if the equations be considered as *simultaneous* their locus consists of all points whose coördinates satisfy both equations — i.e. of all points lying on both surfaces. Hence:

The locus of two simultaneous equations is a curve, viz. the *curve of intersection of the surfaces represented by the two equations separately.*

It is obvious that an infinite number of surfaces may be drawn, all intersecting in the same curve: as a simple example, infinitely many planes may be passed through a given straight line. Thus, while there is only one equation representing a given surface, *a curve may be represented by an infinite number of different pairs of equations* — viz. by the equations of any two of the surfaces having that curve as their intersection.

149. Determination of points on a curve; piercing points. Usually a curved line in space cannot be clearly represented in either form or position by the method of point-plotting*; further, the algebraic work involved in computing the coördinates of a large number of points is very tedious. However, it is often desirable to determine a number of points on a curve; to do this, we *assign values to one of the variables and solve the resulting simultaneous equations for the values of the other two.*

In particular, the *piercing points* of a curve in the coördinate planes are found by *letting x, y, and z in turn equal* 0.

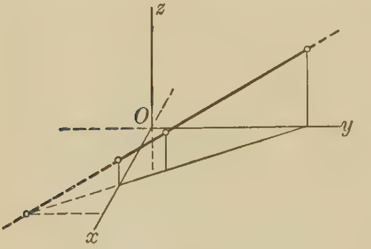


FIG. 116

Example: Determine the piercing points in the coördinate planes and a number of other points of the curve

$$2x + y + z = 3,$$
$$5x + 4y - 2z = 6.$$

When $x = 0$, we have

$$y + z = 3,$$
$$4y - 2z = 6,$$

whence

$$y = 2, \ z = 1.$$

Similarly, the piercing points in the other planes are found to be $(\frac{4}{3}, 0, \frac{1}{3})$, $(2, -1, 0)$. Other points may be determined by assigning values to one of the variables; for instance, when $x = 1$, we find $y = \frac{1}{2}$, $z = \frac{1}{2}$. The curve is a straight line.

---

* An effective method will be presented in § 200.

150. Intersection of three surfaces; of a curve and a surface. It may happen that three surfaces intersect in a curve: for instance, three spheres may have a common circle of intersection. Again, there may be no points common to the three surfaces, as would be the case if two of the three surfaces were concentric spheres. Usually, however, three surfaces intersect in one or more distinct points, real or imaginary. Since these points lie on all three surfaces, their coördinates must satisfy all three of the corresponding equations; the points of intersection of three surfaces are therefore found by *solving their equations as simultaneous.*

A curve and a surface intersect in general in one or more points. If two equations be thought of as representing a curve while the third represents a surface, the points of intersection are found by solving the three simultaneous equations. These are of course the same points as found for the three surfaces represented by the three equations separately; the only difference is in the point of view.

EXERCISES

Determine whether the given points lie on the given curve.

1. Curve $3x - y + 6z = 6$, $x + y + 2z + 2 = 0$, points (*a*) $(3, -3, -1)$, (*b*) $(1, 4, 2)$, (*c*) $(-1, 3, 2)$.

2. Curve $x^2 + y^2 + z^2 = 81$, $2x - y + z = 1$, points (*a*) $(4, 8, 1)$, (*b*) $(-4, -8, -1)$.

3. Curve $x^2 + y^2 = 1$, $z = y + 1$, points (*a*) $(-1, 1, 2)$, (*b*) $(1, 0, 0)$.

4. Curve $x^2 + y^2 = 25$, $x^2 + y^2 + z^2 = 50$, points (*a*) $(5, 0, 0)$, (*b*) $(-3, -4, -5)$, (*c*) $(0, 7, 1)$.

Determine the piercing points in the coördinate planes and one or two other points of each of the following curves.

5. $x - 3y + 2z = 5, 2x + 3y + 2z = 3$.

6. $x + y + 2z = 3, 3x - y - 2z = 1$.

7. $z = x^2,\ z = y^2$.

8. $2x^2 + y^2 + z^2 = 10,\ x^2 + y^2 + z = 8$.

9. $x^2 + y^2 = 10,\ x + 3y + 6 = 0$.

Find the points of intersection of the following loci.

10. Surfaces $x = 1,\ 3x + y + z = 1,\ x - y - 2z = 2$.

11. Surfaces $3x + 2z = y,\ x + 2y + z = 5,\ 2x - 3y + 6 = 2z$.

12. Surfaces $6x + y - z = 1,\ 2x + 5y + 2z + 6 = 0,\ 3x - 3y = z$.

13. Surface $x^2 + y^2 + z^2 = 1$, curve $2x - y + 9z = 1,\ 3x + 6z = 2$.

14. Surface $x + z = 1$, curve $y = x^2,\ y = 4z^2$.

15. Surfaces $y = 2x^2,\ x^2 + z^2 = 2,\ y + 2z = 0$.

16. Surfaces $x^2 + y^2 + z^2 = 2,\ x^2 + y^2 = 1,\ 3x^2 + 3y^2 + 2z^2 = 5$.

151. The surface as locus of a moving point. If a point moves in space subject to a given condition on its coördinates, its locus is in general a certain surface. For, if the coördinates of the moving point in a general position are (x, y, z), the result of expressing analytically the law of motion is an equation in x, y, z which must represent a surface.

Example: A point moves so as to remain equidistant from the points (0, 0, 0), (1, 1, 1). Find the equation of its locus.

If $P: (x, y, z)$ is the moving point, we have

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2},$$

or

$$2x + 2y + 2z = 3.$$

The locus is of course the plane bisecting at right angles the line joining the given points.

152. The curve as locus of a moving point. From the fact that two equations involving its coördinates do not fix the position of a point in space but merely determine a curve in which the point must lie, there arises in solid geometry a type of problem having no analogue in the plane: the type in which a point moves subject to two conditions. The two conditions when expressed analytically give two equations in the coördinates, which taken simultaneously represent the curve which the point describes.

EXERCISES

Find the equation of the locus in each of the following cases.

1. A point moves so as to remain equidistant from the points $(2, 3, -1)$, $(-3, -2, -2)$. *Ans.* $10x + 10y + 2z + 3 = 0$.

2. A point moves so as to remain equidistant from the points $(-\frac{2}{3}, \frac{3}{2}, -\frac{1}{3})$, $(\frac{7}{4}, \frac{5}{4}, -\frac{1}{2})$.

3. A point is equidistant from the xy- and yz-planes.

4. A point lies at the distance $\sqrt{5}$ from the point $(2, -3, 6)$.

5. A point is equidistant from the planes $x = 1$, $y = 2$.
Ans. $x - y + 1 = 0$, and $x + y = 3$.

6. A point moves at the distance 2 from the z-axis.

7. A point remains equidistant from the yz-plane and the point $(4, 0, 0)$. *Ans.* $y^2 + z^2 = 8x - 16$.

8. A point is equidistant from the z-axis and the plane $y = 1$.

9. Solve Ex. 8 in cylindrical coördinates. *Ans.* $r = \dfrac{1}{1 + \sin\theta}$.

Find the equations of each of the following loci.

10. A point moves so as to remain at the distance 3 from the point $(1, 2, 1)$ and at the distance 2 from $(2, 0, 1)$.

11. A point is equidistant from the points $(1, 3, 2)$, $(0, 0, 1)$ and also from the points $(3, 0, 3)$, $(0, -2, 0)$.

12. A point moves at the distance 1 from the x-axis and at the same distance from the y-axis.

13. A point is equidistant from the points $(2, 1, 1)$, $(0, 0, 0)$, $(0, 2, 0)$.

14. A point moves so as to remain equidistant from the x-axis, the yz-plane, and the point $(1, 1, 0)$.

15. Find the equation of a right circular cylinder of radius 3 whose axis is the x-axis.

16. Find the equation of a right circular cylinder of radius 1 whose axis is the line $x = 2$, $z = 1$.

17. The base of an isosceles triangle joins the points $(2, 5, 2)$, $(1, -3, 3)$. Find the locus of the vertex.

CHAPTER XII

THE PLANE

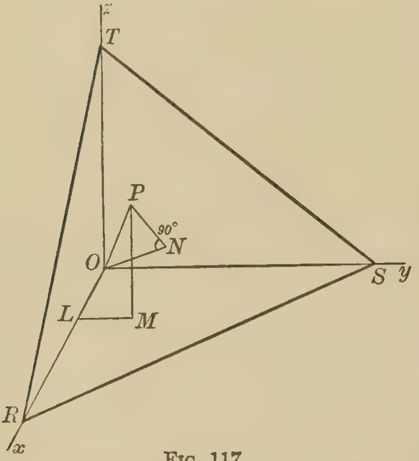


Fig. 117

153. Normal form. Let $P: (x, y, z)$ be any point of the plane RST, and N the foot of the perpendicular from O upon the plane. Denote the length of the normal ON by p and its direction cosines by l, m, n. The line NP is perpendicular to the normal ON if and only if P is in the plane: thus if we can express the perpendicularity of these lines analytically, the result must be the equation of the plane. By (**6**), § 138, the coördinates of N are (lp, mp, np), so that by § 139 the direction cosines of NP are proportional to $x - lp$, $y - mp$, $z - np$. By the corollary of § 142, the lines are perpendicular if and only if

$$l(x - lp) + m(y - mp) + n(z - np) = 0;$$

this is therefore the equation of the plane. Removing parentheses and transposing we get

$$lx + my + nz = l^2p + m^2p + n^2p,$$

or

$$(1) \qquad \boldsymbol{lx + my + nz = p.}$$

This is called the *normal form* of the equation of the plane. For definiteness, the convention is adopted that p shall be always positive.

Even if the plane passes through the origin, the direction cosines l, m, n still serve to determine the direction of any line normal to the plane, so that the derivation still holds. Hence, since formula (**1**) applies to any plane, no matter how situated, we have proved the

THEOREM: *The equation of a plane is always of the first degree.*

154. General form; reduction to normal form. Every equation of the first degree in three variables can be written in the form

$$(2) \qquad \boldsymbol{Ax + By + Cz + D = 0.}$$

Let us transpose the constant term to the right member and make it positive (by changing all signs if necessary), and then divide through by $\sqrt{A^2 + B^2 + C^2}$:

$$(1) \qquad \pm \frac{A}{\sqrt{A^2 + B^2 + C^2}}x \pm \frac{B}{\sqrt{A^2 + B^2 + C^2}}y \pm \frac{C}{\sqrt{A^2 + B^2 + C^2}}z = \frac{\mp D}{\sqrt{A^2 + B^2 + C^2}}.$$

Now the coefficients of x, y, and z are the direction cosines of a certain line, since the sum of their squares is unity: it follows that (1) is *the equation of a plane in the normal form*, the line just referred to being a normal to the plane.

From the fact that this reduction can always be carried out (since $\sqrt{A^2 + B^2 + C^2}$ cannot be 0) we deduce the

THEOREM: *Every equation of the first degree represents a plane.*

Further, we have the

RULE: *To reduce the equation of any plane*

$$Ax + By + Cz + D = 0$$

to the normal form, divide by $\sqrt{A^2 + B^2 + C^2}$ *and choose signs so that the constant is positive in the right member.*

Example: The equation

$$x - y - 2z + 4 = 0$$

becomes in the normal form

$$\frac{-x}{\sqrt{6}} + \frac{y}{\sqrt{6}} + \frac{2z}{\sqrt{6}} = \frac{4}{\sqrt{6}}.$$

The direction cosines of the normal to this plane are therefore

$-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$, and the distance from the origin is $\frac{4}{\sqrt{6}}$.

155. Perpendicular line and plane. From the fact that the coefficients A, B, C in the general equation of the plane are proportional to the direction cosines of the normal, we obtain at once the following

THEOREM: *If a line and plane are perpendicular, the direction cosines of the line and the coefficients of x, y, z in the equation of the plane are proportional; and conversely.*

On account of its numerous applications, this theorem is one of the most important of solid geometry. A typical application is furnished by the following

Example: Write the equation of a plane perpendicular to the radius vector of the point (3, 1, 2) and passing through (0, 4, 5).

The direction cosines of the perpendicular line are proportional to 3, 1, 2; those numbers may therefore be taken as the coefficients in the equation of the plane. The result may be written down at once:

$$3x + y + 2z = 14,$$

the right member being necessarily the value assumed by the left member when the coördinates (0, 4, 5) are substituted.

EXERCISES

1. Write the equation of a plane whose distance from the origin is 2, if the direction cosines of the normal are proportional to 2, 2, -1.
Ans. $2x + 2y - z = \pm 6$.

2. Find the equation of a plane whose distance from the origin is 3, if the normal makes an angle of 60° with Oy and 120° with Oz.
Ans. $\pm \sqrt{2}x + y - z = 6$.

3. Find the equation of a plane through the origin, if the direction cosines of the normal are proportional to 5, -1, 3.

4. Derive the normal form from the fact that the projection of the broken line $OLMPN$ upon the normal must equal ON. Why is this true?

5. Derive the normal form from the fact that the midpoint of OP is equidistant from O and N. Does the derivation hold in all cases?

6. Derive the normal form by applying the theorem of Pythagoras to the triangle ONP. Does the derivation hold in all cases?

7. Find the coördinates of the point N' such that N (Fig. 117) is the midpoint of ON'.

8. Using the result of Ex. 7, derive the normal form by finding the locus of points equidistant from O and N'. Does the derivation hold in all cases?

9. Derive the normal form from the fact that $\cos \phi = \dfrac{ON}{OP}$, where ϕ is the angle between ON and OP.

Reduce the following equations to the normal form; determine the direction cosines of the normal and the distance from the origin. Draw the traces of each plane.

10. $2x + y + 2z = 3$.

11. $3x - 2y + z + 3 = 0$.

12. $3x - 4z = 8y$.

13. $x - y = 6$.

14. $3y = 4$.

15. $2z + 5 = 0$.

Find the equations of the following planes. Sketch each plane by its traces.

16. Passing through the origin, the normal making equal angles with the axes.

17. Through (1, 4, 2) perpendicular to the radius vector of that point.
Ans. $x + 4y + 2z = 21$.

18. Through (2, -1, 3) perpendicular to the radius vector of that point.

19. Having the point $(-1, 6, 3)$ as the foot of the normal from the origin.

20. Having the point $(1, 0, 2)$ as the foot of the normal from the origin.

21. Through $(-3, 5, 4)$ perpendicular to the radius vector of $(2, -4, 3)$.

22. Through the origin perpendicular to the radius vector of $(0, 3, 0)$.

23. Perpendicular to the radius vector of $(1, -2, -2)$ and passing at a distance 2 from the origin.

24. Perpendicular to the line joining $(4, 3, 3)$ and $(-1, -3, 2)$, and passing through the origin. *Ans.* $5x + 6y + z = 0$.

25. Through $(-4, -3, -1)$ perpendicular to the line joining that point and $(2, 2, 3)$.

26. Through $(2, 4, 3)$ perpendicular to the line joining $(1, 2, 6)$ and $(2, 2, -5)$.

27. Perpendicular to the line from $(0, 0, 3)$ to $(1, 3, -4)$ at its midpoint. Solve by two methods.

28. Perpendicular to the line from $(1, 3, 1)$ to $(2, 4, 0)$, and passing at a distance 3 from the origin.

29. The base of an isosceles triangle joins the points $(4, 3, 3)$, $(2, 5, 6)$. Find the locus of the vertex by two methods.

30. One side of a right triangle joins the points $(1, 0, 2)$, $(3, 4, 0)$, the right angle being at the latter point. Find the locus of the third vertex.

156. Parallel planes. If the equations of two planes differ only in the constant term, the same must be true after reduction to the normal form. Hence the normals to the two planes have the same direction cosines and are parallel, from which it follows that *the planes are parallel.*

Conversely, *if two planes are parallel, their equations can be made to differ only in the constant term.*

Example: Write the equation of a plane through $(1, 2, 1)$ parallel to the plane

$$2x + 2y + z = 3.$$

The answer is

$$2x + 2y + z = 7,$$

the right member being written down immediately from the fact that it must be the value assumed by the left member when the coördinates (1, 2, 1) are substituted.

157. Distance between parallel planes; between a point and a plane. The distance between two parallel planes is found by reducing both equations to the normal form and taking the difference or sum of the p's, according as the planes are on the same or opposite sides of the origin.

The distance of a point from a plane is found by writing the equation of a plane through the point parallel to the given plane and then finding the distance between the parallel planes.

EXERCISES

Find the equations of the following planes; draw the figures.

1. Through (3, 1, 1) parallel to the plane $2x + 3y + z = 1$.

2. Through (0, − 2, 3) parallel to the plane $4x - y + 2z = 3$.

3. Through (4, − 2, − 2) parallel to the plane $2y - 3z + 1 = 0$.

4. Through (4, 2, − 1) parallel to the xy-plane.

5. Through (1, − 3, 2) perpendicular to the x-axis.

6. Parallel to the plane $3x - y + z = 6$ and (a) twice as far from the origin; (b) at the same distance from the origin; (c) at the distance 2 from the origin. *Ans.* (c) $3x - y + z = \pm 2\sqrt{11}$.

7. Parallel to the plane $x - 2y + 2z + 6 = 0$ and (a) 2 units farther from the origin; (b) 1 unit nearer the origin; (c) at a distance 2 from the origin.

8. Parallel to the plane $8x - y - 4z + 3 = 0$ and (a) at a distance 3 from that plane; (b) passing at a distance 2 from the origin; (c) having the x-intercept 4.

9. Parallel to the plane $x - y - z + 5 = 0$ and passing at the distance $2\sqrt{3}$ from (1, 2, − 2). *Ans.* $x - y - z = 1 \pm 6$.

10. Parallel to the plane $2x - 3y - 5z + 1 = 0$ and passing at the distance 2 from (− 1, 3, 1).

11. Find the distance between the planes $3x - y + 2z + 6 = 0$, $3x - y + 2z + 5 = 0$.

12. Find the distance between the planes $x + y + 2z + 8 = 0$, $x + y + 2z - 4 = 0$.

13. A sphere touches the planes $2x + 4y + 6z = 3$, $x + 2y + 3z + 1 = 0$. Find its volume and surface, and the locus of its center.

14. A sphere of radius $2\sqrt{6}$ rolls on the plane $2x - y - z = 2$. Find the locus of its center.

15. Two faces of a cube lie in the planes $x + 2y + 2z = 3$, $3x + 6y + 6z + 1 = 0$. Find the volume of the cube.

16. A sphere of radius 3 touches the plane $2x - y + 3z = 1$. Find the locus of its center.

In each of the following cases, find the distance of the point from the plane.

17. $2x - y + 2z + 6 = 0$, $(1, -3, 1)$. *Ans.* $\frac{13}{3}$.

18. $z - 8x - 9y = 2$, $(2, -3, 0)$. *Ans.* $\dfrac{9}{\sqrt{146}}$.

19. $x - 5y + 3z + 1 = 0$, $(4, 1, 0)$.

20. $3x + 7 = 0$, $(2, -1, 3)$.

21. $3x - 4z = 6$, $(\frac{3}{5}, \frac{2}{5}, \frac{3}{10})$.

22. A sphere with center at $(1, 5, 2)$ touches the plane $x + 4y + 8z = 18$. Find the volume and surface of the sphere; also find the equation of another tangent plane parallel to the given one.

158. Plane through a given point. If the plane

$$Ax + By + Cz + D = 0 \tag{1}$$

is to pass through the point (x_1, y_1, z_1), those coördinates must satisfy the equation, and we have

$$Ax_1 + By_1 + Cz_1 + D = 0. \tag{2}$$

Elimination of D between (1) and (2) gives the following useful result:

The equation of any plane through the point (x_1, y_1, z_1) *may be written in the form*

$$\mathbf{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.} \tag{3}$$

159. Plane determined by three points. From the fact that the general equation of the plane contains three essential constants — viz. the ratios of any three of the quantities

A, B, C, D to the fourth — we conclude that *a plane is determined by three points*, or by any set of three conditions which when expressed analytically give three equations to determine the essential ratios.

The direct method of finding the equation of a plane determined by three points is to substitute the coördinates of the points in turn in the equation

$$Ax + By + Cz + D = 0,$$

thus obtaining three equations to solve for three of the constants in terms of the fourth. The solution may, however, be expedited by use of the formula of § 158, as in the following

Example: Find the equation of the plane through the points $(0, -1, 1)$, $(-1, 3, 2)$, $(3, 2, -3)$.

The equation of any plane through $(0, -1, 1)$ is

(1) $$Ax + B(y + 1) + C(z - 1) = 0.$$

Substituting the coördinates of the other points in (1), we get

(2) $$-A + 4B + C = 0,$$

$$3A + 3B - 4C = 0.$$

Elimination of A gives

$$C = 15B;$$

substituting this value in (2) we find

$$A = 19B.$$

Hence the equation of the plane is found by substitution in (1) to be

$$19Bx + B(y + 1) + 15B(z - 1) = 0,$$

or

$$19x + y + 15z - 14 = 0.$$

Check. $$0 - 1 + 15 - 14 = 0,$$

$$-19 + 3 + 30 - 14 = 0,$$

$$57 + 2 - 45 - 14 = 0.$$

160. Angle between two planes. The angle between two planes is evidently equal to the angle between their normals. Hence the angle between two planes is found by substituting the direction cosines of the normals in the formula for the angle between two lines (§ 141).

161. Perpendicular planes. Two planes

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0$$

are perpendicular if their normals are perpendicular to each other. The direction cosines of the normals are proportional to A_1, B_1, C_1 and A_2, B_2, C_2 respectively. Applying the condition of perpendicularity (Corollary, § 142) to these two lines, we obtain the following

THEOREM: *Two planes*

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0$$

are perpendicular if

$$\mathbf{(4)} \qquad \boldsymbol{A_1A_2 + B_1B_2 + C_1C_2 = 0;}$$

and conversely.

Example: Find the equation of a plane through the points (1, 1, 2), (2, 4, 3), and perpendicular to the plane

$$(1) \qquad x - 3y + 7z + 5 = 0.$$

The equation of a plane through (1, 1, 2) is

$$(2) \qquad A(x - 1) + B(y - 1) + C(z - 2) = 0.$$

Substituting the coördinates (2, 4, 3) in (2), we get

$$(3) \qquad A + 3B + C = 0.$$

The condition of perpendicularity (4) applied to the planes (1) and (2) gives

$$(4) \qquad A - 3B + 7C = 0.$$

From (3) and (4) we find

$$A = -4C, B = C,$$

whence the equation of the plane is

$$-4C(x-1)+C(y-1)+C(z-2)=0,$$

or

$$4x-y-z=1.$$

Check:
$$4-1-2=1,$$
$$8-4-3=1,$$
$$4+3-7=0.$$

162. Planes perpendicular to a coördinate plane. An equation of the first degree in which x is missing, say

$$By+Cz+D=0,$$

represents a plane perpendicular to the yz-plane. For the equation of the yz-plane is

$$x=0,$$

and the condition of perpendicularity (4), § 161, applied to these two planes gives

$$0\cdot 1+B\cdot 0+C\cdot 0=0,$$

which is true for all values of B and C. A similar result holds for equations in which y or z is missing. Hence the

THEOREM: *An equation of the first degree in two variables represents a plane perpendicular to the plane of those two variables; and conversely.*

EXERCISES

Find the equations of the following planes.

1. Through $(0, 0, 0)$, $(1, 1, -2)$, $(-1, 1, 4)$.

2. Through $(2, 2, 2)$, $(3, 1, 1)$, $(3, -4, -9)$.

3. Through $(3, 3, 1)$, $(-3, 2, -1)$, $(8, 6, 3)$.

4. Through $(0, 1, 1)$, $(2, 4, 3)$, $(3, -5, 6)$.

5. Through $(1, -2, 2)$, $(0, 4, -5)$, $(-2, 1, 0)$.

6. Through $(2, 3, -3)$, $(1, 0, -5)$, $(3, 6, -1)$. Explain the meaning of the result.

7. Find the angle between the planes $x+2y-z=6$, $x-2y-2z=3$. *Ans.* Arccos $\frac{1}{18}\sqrt{6}$.

8. Find the angle between the planes $3x - z + 2 = 0$, $x + 3y + 5 = 0$.

9. Find the angle between the planes $y + z = 3x$, $x - 2y = 0$.

10. Find the angle between the plane $x + 3y + 4z = 6$ and the xy-plane.

Find the equations of the following planes.

11. Through (1, 2, 3), (3, − 1, − 19), perpendicular to the plane $3x + 2y + 6z + 4 = 0$. *Ans.* $2x - 6y + z + 7 = 0$.

12. Through (0, 2, 3), (5, − 1, 4), perpendicular to the plane $x + 2y + 3 = 0$.

13. Through (− 1, 3, − 1), (− 5, 4, 2), perpendicular to the zx-plane. *Ans.* $3x + 4z + 7 = 0$.

14. Through (1, 1, 5), perpendicular to the planes $3x - y + 4z = 5$, $6x - 2y + 5z = 7$. *Ans.* $x + 3y = 4$.

15. Through the origin, perpendicular to the planes $x + y + 2z = 3$, $2x - y - 4z = 6$.

16. Through (2, 1, 1), (3, 2, 2), perpendicular to the plane $x + 2y - 5z = 3$.

17. Through (1, 2, 2) perpendicular to the xy- and yz-planes.

18. Perpendicular to the planes $3x + 4y - z = 5$, $6x + 5y = 2z$, and passing at a distance $\frac{1}{2}\sqrt{10}$ from O. *Ans.* $x + 3z = \pm 5$.

19. Through (1, 2, 1), parallel to the x-axis, and passing at a distance 1 from the origin.

20. Through (3, 3, 2), perpendicular to the xy-plane and to the plane $2x - 3y = 6$.

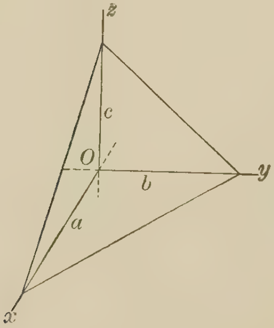


FIG. 118

163. Intercept form. If a, b, c denote the intercepts of a plane on the x-, y-, and z-axes respectively, the equation of the plane can be found from the fact that it passes through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. The derivation is left to the student; the result is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \tag{5}$$

as is easily verified.

164. Intersection of three planes. Since three simultaneous equations of the first degree in three unknowns have in general one and only one solution, it follows that three planes intersect in general in one point. However, they may have a common line of intersection, in which case the values of x, y, and z found by solving the simultaneous equations are indeterminate. Again, two of the planes may be parallel, or the three planes may bound a triangular prism, in which cases the equations are incompatible.

EXERCISES

Reduce the following equations to the intercept form, and draw the planes.

1. $4x + y + 2z = 6$. **2.** $x - y + 3z = 1$.

3. $x - 2y + 2z + 5 = 0$. **4.** $4x + 3y + 6z + 8 = 0$.

5. In what cases does the intercept form fail?

6. Derive the intercept form as suggested in § 163.

7. A plane makes equal intercepts on the axes and passes through (3, 4, 2). Find its equation.

8. A plane makes equal intercepts on the axes and passes at a distance 1 from the origin. Find its equation.

9. The traces of a plane are, in the yz-plane, $3y + 2z = 4$, in the zx-plane, $x + 3z = 6$. Find the equation of the plane.

10. Find the equation of the plane two of whose traces are $x + 3y + 5 = 0$, $2x - z + 10 = 0$.

Investigate the nature of the intersections of the following sets of planes.

11. $x - 3y + 2z = 6$, $3x - y - 5z + 4 = 0$, $2x - y + 2z = 2$.

12. $x + y + z = 2$, $4x - 2y - z + 1 = 0$, $3x + 2y + z = 8$.

13. $3x - y - 2z + 1 = 0$, $x + 3y + 4z = 5$, $6x - 2y - 4z + 3 = 0$.

14. $2x + y + 5z = 4$, $x - 3y + z = 5$, $x + 4y + 4z + 1 = 0$.

15. $5x + 8y - 9z + 10 = 0$, $3x - 11y - 6z + 5 = 0$, $x + 4y + 8z = 8$.

16. $3x + 2y + 6 = 0$, $x - y - 3z + 2 = 0$, $6x - y - 9z = 0$.

17. $3x - y + z = 8$, $x + 4y - 2z + 2 = 0$, $2x - 3y - z = 2$.

18. $y = 3x + 4$, $6x - 2y = 7$, $x + 5z = 0$.

CHAPTER XIII

THE STRAIGHT LINE

165. General form. From the fact that two planes intersect in a straight line, we deduce the

THEOREM: *The locus of two simultaneous equations of the first degree is a straight line.*

Thus the general form of the equations of a straight line is

$$(1)\qquad \begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

Since through any line an infinite number of planes may be passed, it is obvious that a straight line may be represented by two simultaneous equations in an infinite number of ways — viz. by the equations of any two planes through the line. Of these various pairs of equations there are usually one or two especially simple and convenient; we will show in the next few articles how to derive the simpler forms from the general form.

166. Planes through a given line. Let the equations of a straight line be denoted by

$$(1)\qquad \begin{cases} u = 0, \\ v = 0, \end{cases}$$

where u and v represent any expressions of the first degree in x, y, and z. Then the equation

$$(2)\qquad u + kv = 0,$$

where k is a constant, is an equation of the first degree and thus represents a plane; further, since (2) is satisfied whenever both the equations (1) are satisfied, the plane (2) contains all the points of the line (1). (The student should compare this reasoning with that of § 57.) Hence we have the

THEOREM: *If*

$$u = 0,\ v = 0$$

are the equations of any line, the equation

$$u + kv = 0, \tag{2}$$

where k is a constant, represents a plane through the given line.

It is geometrically obvious that a plane containing a given line may be made to pass through a given point, or to be perpendicular to another plane, or to pass at a given distance from a fixed point, etc. The truth of these statements appears analytically from the fact that equation (**2**) contains one undetermined constant.

Any two equations of the family (**2**) may of course be used as equations of the line of intersection, instead of the original pair.

167. Projecting planes. The planes through a line perpendicular to the coördinate planes are called the *projecting planes*, and their traces are the *projections*, of the line on the coördinate planes. It is often convenient, for instance in making a sketch, to represent a line by the equations of two of its projecting planes.

If in (**2**), § 166, we choose k so that the term in x drops out, which amounts merely to eliminating x between the equations of the line, the resulting equation represents a plane through the given line perpendicular to the yz-plane (§ 162). In this way we establish the

RULE: *To obtain the yz-projecting plane of a line, eliminate x between the equations of the line; similarly for the other projecting planes.*

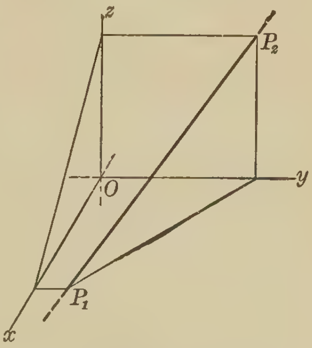


FIG. 119

In Fig. 119 the line P_1P_2 is exhibited as the intersection of its xy- and zx-projecting planes.

EXERCISES

Find the piercing points of the following lines in the coördinate planes, and draw the lines.

1. $x + 3y + z = 7$, $3x + 5y - z = 9$.
Ans. (0, 2, 1), (4, 0, 3), (– 2, 3, 0).

2. $8x - 3y + 4z = 20$, $6x - 3y + 2z = 12$.

3. $x + 2y + 3z = 6$, $3x - y - z = 4$.

4. $x + 3y + 5z = 4$, $x + 6y + 10z = 10$.

5. $3x + y = 5$, $2x - 3y = 7$.

6. $x + 2y + 2z = 10$, $2x - y + 4z = 5$.

7. Write the equations of a line through (1, – 3, 2) parallel to the line $3x + 2y + 2 = 0$, $x - y - z = 3$.

8. Write the equations of a line through (3, – 1, 4) parallel to the y-axis.

Find the equations of the following planes.

9. Through the line $x + y = 1$, $z = 4$ and the point (2, 3, 0). Draw the figure. *Ans.* $x + y + z = 5$.

10. Through the line $x + y - z = 3$, $x + 2y - z = 6$ and the point (0, 0, 0). Check by a sketch. *Ans.* $x - z = 0$.

11. Through the line $3x + y - z = 3$, $x + 2y + 4z + 4 = 0$ and the point (1, 1, – 2). *Ans.* $6x + 7y + 11z + 9 = 0$.

12. Through the x-axis and the point (1, 3, 2). *Ans.* $2y = 3z$.

13. Through the line $3x + y - z = 3$, $x + 2y + 4z + 4 = 0$, perpendicular to the plane $x - 2y - z = 5$. Draw the figure.
Ans. $23x + 11y + z = 13$.

14. Through the line $x - 2y - 3z + 4 = 0$, $2x - y + 4z = 6$, perpendicular to the xy-plane.

15. Perpendicular to the plane $x + 3y - z = 1$, and having as its yz-trace the line $y + 2z = 3$. *Ans.* $x - y - 2z + 3 = 0$.

16. Through the line $x = y = z$, perpendicular to the plane $x + y + 2z = 0$.

17. Show that the planes $2x + 5y + 3z = 0$, $7y - 5z + 4 = 0$, $x - y + 4z = 2$ intersect in a line.

18. Show that the line $3x + y + z = 6$, $x - 2y + 4z = 5$ lies in the plane $7x + 7y - 5z = 8$.

19. Find the equations of a line in the plane $4x + 2y + z = 10$, through the point $(1, 1, 4)$, and parallel to the xy-plane. Draw the figure.

20. Find the equations of a line in the plane $3x - y = 6$, through the point $(3, 3, 1)$, and parallel to the plane $5x + 6y + 3z = 15$. Draw the figure.

Find the projecting planes of the following lines, and draw the lines.

21. $x + y + 3z = 3,\ 2x + y + 4z = 4$.

22. $4x + y + z = 4,\ 3x + 3y + 2z = 6$.

23. $x - y + z = 0,\ 3x - y = 0$.

24. $x + y = 1,\ 3x - y - z = 2$.

25. $x + y + z = 3,\ x + y + 3z = 5$.

26. $x + 2z = 5,\ 3x - z = 1$.

27. $3x + y - 2z = 0,\ x + y - z = 0$.

168. Symmetric form. The equations of the line through (x_1, y_1, z_1) having direction cosines proportional to three numbers a, b, c may be found as follows. If (x, y, z) is any point of the line, the direction cosines are proportional to $x - x_1, y - y_1, z - z_1$. Since two sets of numbers proportional to the same third set are proportional to each other, the numbers $x - x_1, y - y_1, z - z_1$ are proportional to a, b, c: that is,

$$\textbf{(3)} \qquad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Since these two equations involve the coördinates x, y, z of any point of the line, and no other variables, they are the equations of the line. Form (**3**) is called the *symmetric form;* for most purposes it is more convenient than any other.

It may be objected that in (**3**) there are three equations, viz.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}, \quad \frac{x - x_1}{a} = \frac{z - z_1}{c}, \quad \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

These equations, however, are not independent, since any one follows from the other two. These equations of course represent three planes through the line; they are in fact the projecting planes.

As a simple application of the symmetric form we may cite the following

Example: Write the equations of the line joining the points (2, 3, − 1), (1, − 3, 1).

By § 139, the direction cosines are proportional to 1, 6, − 2; hence the equations of the line may be written in the form

$$\frac{x-2}{1}=\frac{y-3}{6}=\frac{z+1}{-2}.$$

169. Determination of direction cosines; reduction to the symmetric form. Being given the equations of a line in the general form, it is often necessary to reduce them to the symmetric form, or at least to find numbers proportional to the direction cosines. This may be done by finding two points of the line and then applying the theorem of § 139.

Example: Reduce the equations

$$x+3y+z=7,$$
$$3x+5y-z=9$$

to the symmetric form.

In Ex. 1, p. 244, two points of this line were found to be P_1: (0, 2, 1), P_2: (4, 0, 3). Hence the direction cosines are proportional to 4, − 2, 2, or 2, − 1, 1; the actual values of the cosines are $\frac{2}{\sqrt{6}}$, $\frac{-1}{\sqrt{6}}$, $\frac{1}{\sqrt{6}}$; and the equations of the line may be written in the symmetric form as

$$\frac{x-4}{2}=\frac{y-0}{-1}=\frac{z-3}{1}.$$

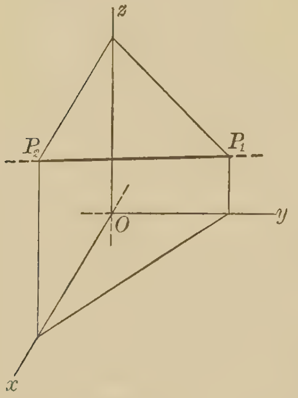


FIG. 120

170. Reduction to the symmetric form: second method. A somewhat shorter method for reducing the equations of a line to the symmetric form is as follows:

Find the equations of two of the projecting planes. These equations will in general have one variable in common; solve both equations for this variable and equate the values thus found. Certain obvious transformations then bring the desired result. The process is best explained in detail by an

Example: Reduce the equations

$$x + 3y + z = 7, 3x + 5y - z = 9$$

to the symmetric form.

By eliminating y and z in turn, the zx- and xy-projecting planes are found to be

$$x - 2z = -2,$$
$$x + 2y = 4,$$

or, after solving for the common variable x, and equating,

$$x = -2y + 4 = 2z - 2,$$

whence

$$\frac{x}{2} = \frac{y-2}{-1} = \frac{z-1}{1}.$$

This agrees with the result of § 169.

171. Line parallel to a coördinate plane. If the line is parallel to a coördinate plane, the method of § 170 fails, since the equations of the projecting planes do not contain one variable in common. However, the process is readily modified to take care of this case, as may be shown by an

Example: Reduce the equations

$$2x + 2y + 3z = 8,$$
$$x + 4y + 6z = 10$$

to the symmetric form.

The projecting planes are found to be

(1) $$x = 2,$$
(2) $$2y + 3z = 4.$$

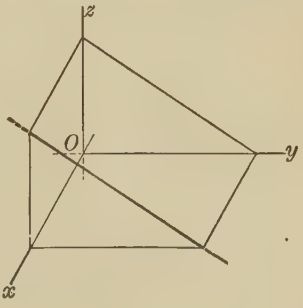


Fig. 121

Equation (1) shows that the line is parallel to the yz-plane, and therefore makes an angle of 90° with Ox. Equation (2) can be put in the form

$$\frac{y}{3}=\frac{z-\frac{4}{3}}{-2}.$$

Hence one point of the line is $(2, 0, \frac{4}{3})$, while the direction cosines are proportional to $0, 3, -2$: the symmetric form is therefore

$$\frac{x-2}{0}=\frac{y}{3}=\frac{z-\frac{4}{3}}{-2}.$$

The line is shown in Fig. 121.

172. Parallel, intersecting, and skew lines. It is obvious that two lines in space do not in general lie in the same plane, in which case they are said to be *skew*. However, they may be parallel, or they may intersect.

Being given the equations of any line in the symmetric form, we may write down at once the equations of the parallel line through any point; for the denominators a, b, c in the equations of the required line must be proportional to, and may be taken equal to, the corresponding denominators in the given equations.

Being given the equations of two pairs of planes, representing two lines, we may find the point of intersection of any three of the planes: the given lines do or do not intersect according as this point does or does not lie in the fourth plane.

EXERCISES

Write the equations of the following lines.

1. Through $(2, 5, -1)$, $(0, 7, 3)$.

2. Through $(1, 1, -1)$, $(2, 7, -3)$.

3. Through $(-2, 3, -2)$, $(2, 4, 5)$.

4. Through $(4, 3, 1)$, $(5, 3, 3)$.

5. Through $(1, 4, -3)$, $(2, 0, -3)$.

6. Through $(3, 5, 2)$, $(0, 5, 2)$.

7. Through (0, 0, 0) with direction cosines proportional to 1, 3, 4. Draw the line.

8. Through (0, 0, 0) making 45° with Ox and 60° with Oz. Draw the line.

9. Through (3, 1, 2) making the angles arccos $\frac{1}{3}$ and arccos $\frac{1}{4}$ with Ox and Oy respectively.

10. Through $(a, 0, 0)$, $(0, b, 0)$.

11. Derive the intercept form (§ 163) by writing the equation of a plane through the line of Ex. 10 and the point $(0, 0, c)$.

12. Find the angle between the lines

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{-2}, \quad \frac{x-1}{2} = \frac{y-1}{4} = \frac{z}{3}.$$

13. Find the angle between the lines

$$\frac{x}{0} = \frac{y-1}{1} = \frac{z+2}{1}, \quad \frac{x}{3} = \frac{y+1}{1} = \frac{z-3}{0}.$$

14. Show that the lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$, $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z+1}{1}$ are perpendicular.

15. Find the direction cosines of the line $3x + y - z = 1$, $2x + 2y + z = 0$. *Ans.* $\frac{3}{10}\sqrt{2}$, $-\frac{1}{2}\sqrt{2}$, $\frac{2}{5}\sqrt{2}$.

16. Reduce the equations $x + 2y - 3z = 6$, $3x + 4y + z = 5$ to the symmetric form.

17. Find the angle between the lines $x + 2y + 2z + 2 = 0$, $x - 4y - z = 1$ and $x - 2y + z = 0$, $2x - y - z = 0$.
Ans. Arccos $\frac{1}{9}\sqrt{3}$.

18. Reduce the equations $4x - y + z = 1$, $4x - y + 3z = 3$ to the symmetric form. Draw this line.

19. Reduce the equations $y + z = 3$, $3y + 2z = 7$ to the symmetric form. Draw the line. *Ans.* $\frac{x}{1} = \frac{y-1}{0} = \frac{z-2}{0}$.

20. Find the angle between the lines $x + y + 3z = 1$, $2x + 3y + 6z = 3$ and $x - z = 0$, $x + 3y - z = 5$.

21. Write the equations of a line through (0, 2, 3) parallel to the line $\frac{x}{3} = \frac{y+1}{1} = \frac{z}{-4}$.

22. Write the equations of a line through (1, 1, 2) parallel to the line $\frac{x}{1} = \frac{y-2}{2} = \frac{z-5}{1}$.

23. By a new method, find the equations of the line through $(1, -3, 2)$ parallel to the line $3x + 2y + 2 = 0$, $x - y - z = 3$. (Ex. 7, p. 244.)

24. By two methods, determine the line through $(1, 4, -6)$ parallel to the line $x + 3y = 5$, $x = z$.

25. Show that the lines $x + y - z = 0$, $x - 3y + z = 0$ and $3x - y - z = 2$, $4x - 6y + z = 1$ are parallel.

26. Find the equation of the plane determined by the parallel lines of Ex. 25. *Ans.* $5x - 11y + 3z = 0$.

27. Find the equation of the plane determined by the parallel lines $\frac{x}{1} = \frac{y-2}{-2} = \frac{z-1}{1}$, $\frac{x}{1} = \frac{y-3}{-2} = \frac{z}{1}$. Draw the figure.

28. Show that the lines $5x + 7y - 7z = 3$, $3x + 9y - z = 0$ and $x + 2y = z$, $2x - 6z = 2y + 3$ are parallel.

29. Find the equation of the plane determined by the parallel lines of Ex. 28.

30. Find the point of intersection of the lines $2x - 3y + 2z = 0$, $x + y - 3z = 4$ and $3x - y + z = 2$, $7x - 5y + 6z = 1$.

31. Find the equation of the plane determined by the lines of Ex. 30.

32. Find the point of intersection of the lines $x = -2z + 3$, $y = z - 2$ and $x = 3z - 1$, $5y = -10z + 2$.

33. Find the equation of the plane through the lines of Ex. 32.

34. Find the plane through the intersecting lines

$$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-1}{1};\quad \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-1}{-1}.$$

Draw the figure. *Ans.* $2x + y + 3z = 7$.

173. Perpendicular line and plane. As a corollary to the theorem of § 155 we have at once the following

THEOREM: *The plane*

$$Ax + By + Cz + D = 0$$

and the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

are perpendicular if and only if the quantities A, B, C and a, b, c are proportional.

Example: Write the equations of the line through (1, 2, 3) perpendicular to the plane

$$2x - 3y + 5z = 6.$$

The answer is obviously

$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{5}.$$

174. Angle between a line and a plane. The angle ϕ between a line and a plane is evidently the complement of the angle ϕ' between the line and a normal to the plane. If l_1, m_1, n_1 are the direction cosines of the line and l_2, m_2, n_2 those of the normal to the plane, we have by § 141

$$\cos\phi' = l_1l_2 + m_1m_2 + n_1n_2;$$

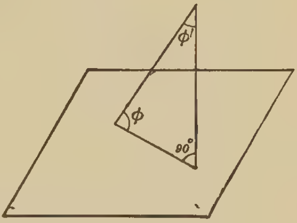


FIG. 122

hence the angle between the line and the plane is given by the formula

$$\sin\phi = l_1l_2 + m_1m_2 + n_1n_2. \tag{4}$$

175. Parallel line and plane. If a line is parallel to a plane, it is perpendicular to the normal to the plane. The direction cosines of the normal are proportional to the coefficients A, B, C in the equation of the plane. Applying the condition for perpendicular lines (Corollary, § 142), we deduce at once the

THEOREM: *The line*

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

and the plane

$$Ax + By + Cz + D = 0$$

are parallel if

$$aA + bB + cC = 0; \tag{5}$$

and conversely.

EXERCISES

1. Write the equations of the line through (2, 5, 1) perpendicular to the plane $2x + y + 3z = 6$. Draw the figure.

2. Write the equations of the line through (– 1, 0, 0) perpendicular to the plane $y - z = 3x + 4$.

3. Write the equations of the line through (2, 1, 2) perpendicular to the plane $y = x$. Draw the figure.

4. Find the foot of the perpendicular from the origin upon the plane $x + y + 2z = 3$. Draw the figure. *Ans.* $(\frac{1}{2}, \frac{1}{2}, 1)$.

5. Find the foot of the perpendicular from the point (2, 3, 6) upon the plane $2y + z = 6$. Draw the figure.

6. Write the equation of the plane through $(\frac{6}{5}, -\frac{3}{2}, \frac{2}{3})$ perpendicular to the line $\frac{x-3}{2} = \frac{y-1}{3} = \frac{z}{6}$.

7. Write the equation of the plane through (8, 5, – 6) perpendicular to the line $\frac{x}{1} = \frac{y}{3} = \frac{z}{0}$.

8. Find the equation of the plane through (1, 5, 3) perpendicular to the line $x + 4y + z = 5$, $2x + 4y - 2z = 3$.
Ans. $3x - y + z = 1$.

9. Solve Ex. 8 by another method.

10. Find the equation of a plane perpendicular to the line $\frac{x}{3} = \frac{y-2}{2} = \frac{z}{4}$ and passing at a distance 2 from the origin.

11. Find the distance of the origin from the line $2x - y + z = 1$, $4x - y + 3z = 3$. Draw the figure. *Ans.* $\frac{1}{3}\sqrt{6}$.

12. Find the equations of the line through (1, 3, 5) intersecting the line $\frac{x}{4} = \frac{y-1}{2} = \frac{z+3}{-1}$ at right angles.

Ans. $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z-5}{8}$.

13. Find the distance of the point (1, 0, 3) from the line $2x - y - z + 1 = 0$, $7x - 2y - 8z + 8 = 0$. *Ans.* $\frac{3}{7}\sqrt{21}$.

14. Find the equations of the line through the origin intersecting at right angles the line $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-3}{3}$.

15. Find the angle between the plane $2x - 4y + 3z = 1$ and the line $2x - 3y + 2 = 0$, $2y - z + 9 = 0$. *Ans.* Arcsin $\frac{10}{29}$.

16. Find the angle between the xy-plane and the line $x+y=4$, $x+z=3$. Draw the figure.

17. Find the angle between the x-axis and the plane $x+y+2z=4$.

18. Find the distance of the point (4, 4, 4) from the line

$$\frac{x}{2}=\frac{y-3}{-1}=\frac{z-4}{-1}.$$

Draw the figure.

19. Find the equation of a plane through the points (1, 1, 5), (2, 3, −2) parallel to the line $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z+3}{-2}$.

Ans. $3x-5y-z+7=0$.

20. Find the equation of a plane through the points (0, 0, 0), (3, 3, 2) parallel to the line $\frac{x}{1}=\frac{y}{0}=\frac{z}{3}$.

21. By various methods, find the equations of a line through (1, −5, −3) parallel to each of the planes $3x+y-z=5$, $x+4y+3z=6$.

22. By two methods, find the locus of points equidistant from the points (1, 4, 0), (2, 3, 2).

23. A point moves so as to remain equidistant from the points (2, 0, 0), (0, 2, 0), (0, 0, 2). Find the equations of its locus, and draw the curve.

24. Find the equation of the plane through the origin parallel to the plane determined by the intersecting lines

$$\frac{x-3}{1}=\frac{y}{2}=\frac{z}{3},\quad \frac{x-3}{2}=\frac{y}{2}=\frac{z}{-3}.$$

Ans. $12x-9y+2z=0$.

25. Show that the plane $x-y+z=3$ and the line

$$\frac{x}{1}=\frac{y+2}{1}=\frac{z-3}{0}$$

are parallel, and find the distance between them. *Ans.* $\frac{2}{3}\sqrt{3}$.

26. In Ex. 25, find the distance by another method.

27. Show that the plane $x+3y+z=6$ and the line

$$\frac{x}{1}=\frac{y-4}{-1}=\frac{z-4}{2}$$

are parallel, and find the distance between them. Draw the figure.

28. Find the equation of a plane through the y-axis parallel to the line $x=y=z-1$. Draw the figure.

29. Find the area of the triangle whose vertices are (0, 0, 0), (2, 2, 2), (3, 4, 2). *Ans.* $\sqrt{6}$.

30. Find the distance between the parallel lines

$$\frac{x}{2} = \frac{y-2}{1} = \frac{z-1}{3}, \quad \frac{x-1}{2} = \frac{y}{1} = \frac{z+2}{3}. \qquad \textit{Ans. } \tfrac{1}{14}\sqrt{1610}.$$

31. Find the distance between the parallel lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}, \quad \frac{x-3}{1} = \frac{y+1}{2} = \frac{z-2}{2}. \qquad \textit{Ans. } \tfrac{1}{3}\sqrt{101}.$$

32. Find the locus of points equidistant from the parallel lines of Ex. 31.

33. Find the shortest distance between the skew lines

$$\frac{x-1}{1} = \frac{y-3}{1} = \frac{z}{-2}, \quad \frac{x}{3} = \frac{y}{5} = \frac{z}{-16}. \qquad \textit{Ans. } \frac{12}{\sqrt{35}}.$$

34. In Ex. 33, find that point of each line that is nearest the other line. *Ans.* $(-\tfrac{15}{7}, -\tfrac{1}{7}, \tfrac{44}{7}), (-\tfrac{39}{35}, -\tfrac{13}{7}, \tfrac{208}{35})$.

35. Find the shortest distance between the skew lines

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{1}, \quad \frac{x-3}{1} = \frac{y}{-1} = \frac{z}{-1}. \qquad \textit{Ans. } \frac{3}{\sqrt{14}}.$$

36. In Ex. 35, find that point of each line that is nearest the other line. Draw the figure. *Ans.* $(\tfrac{45}{28}, \tfrac{45}{28}, \tfrac{15}{28}), (\tfrac{51}{28}, \tfrac{33}{28}, \tfrac{33}{28})$.

CHAPTER XIV

SURFACES

176. Introduction. In studying a surface analytically, usually the first step is to derive the equation from the definition; the next is to present the locus to the eye by a drawing of some kind. These problems will be solved, for some of the more important surfaces, in the present chapter.

Before taking up the study of particular surfaces, we shall in the next three articles develop certain general results that will find frequent application in our subsequent work.

177. Symmetry. Two points P_1, P_2 are said to be *symmetric with respect to a plane* if the plane is perpendicular to the line P_1P_2 at its midpoint — i.e. if P_2 is the image, or reflection, of P_1 in that plane. A geometric figure is symmetric with respect to a plane if corresponding to every point P_1 of the figure the image P_2 also belongs to the figure.

The definitions of *axis* and *center* of symmetry laid down in § 16 apply without change to figures in space.

The student will have no difficulty in establishing the following tests for symmetry with respect to the coördinate planes, the axes, and the origin.

Theorem I: *A surface is symmetric with respect to the yz-plane if x can be replaced by $-x$ without changing the equation; and conversely.* Similarly for symmetry with respect to the zx- and xy-planes.

Theorem II: *A surface is symmetric with respect to the x-axis if y and z can be replaced by $-y$ and $-z$ simultaneously without changing the equation; and conversely.* Similarly for symmetry with respect to the y- and z-axes.

THEOREM III: *A surface is symmetric with respect to the origin if x, y, and z can be replaced by $-x$, $-y$, and $-z$ simultaneously without changing the equation; and conversely.*

In making a complete sketch of a surface, the amateur draftsman is often hindered by mechanical difficulties. Considerations of symmetry frequently reduce or remove these difficulties by making it unnecessary to draw more than a portion of the figure. Thus if a surface is symmetric with respect to the xy-plane, only the part lying above that plane need be drawn; if it is symmetric with respect to all three coördinate planes, only the part lying in the first octant need be drawn.

Example: The surface

$$yz = x^2$$

is symmetric with respect to the yz-plane, the x-axis, and the origin, but not with respect to the other elements of reference.

178. Sketching by parallel plane sections. One of the most effective methods of sketching a surface is by means of a series of plane sections parallel to a coördinate plane. It often happens that one particular set of sections will form a clearer picture of the surface than any other set, so that care should be taken to make the best choice.

Before making the drawing, it is well to carry out the following steps:

(1) *Test the surface for symmetry.*

(2) *Determine the traces on the coördinate planes.*

(3) *Determine the nature of the sections parallel to each coördinate plane.*

Example: Sketch the surface

$$x^2 + y^2 = az.$$

(1) The surface is symmetric with respect to the yz- and zx-planes and the z-axis.

(2) The traces are the parabola $y^2 = az$, the parabola $x^2 = az$, and the point circle $x^2 + y^2 = 0$.

(3) Setting $x = k$, we get as sections parallel to the yz-plane the parabolas

$$y^2 = az - k^2;$$

the sections parallel to the zx-plane are the parabolas

$$x^2 = az - k^2;$$

those parallel to the xy-plane are, when* $k < 0$, imaginary, when $k > 0$, the circles

$$x^2 + y^2 = ak.$$

Evidently the surface is best pictured by means of its traces and circular sections, as in Fig. 123. On account of the symmetry, it is sufficient to draw only the portion lying in the first octant.

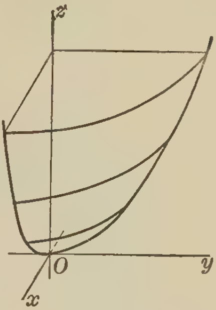


Fig. 123

EXERCISES

Test the following surfaces for symmetry with respect to the coordinate planes, the axes, and the origin.

1. $x^2 + y^2 + z^2 = 25.$

2. $x^2 - z^2 = 2y.$

3. $x + 2y = 6.$

4. $y^2 + z^2 = x^2.$

5. $y^2 + z^2 = 2xy.$

6. $x^2 + z = 3.$

7. $x^2 + y^2 = a^2.$

8. $z = 6.$

9. $xy + yz + zx = 0.$

10. $x^2y = z^3.$

For each of the following figures of elementary geometry, discuss the number and position of the planes and lines of symmetry.

11. A sphere.

12. A right circular cone.

13. A right circular cylinder.

14. A rectangular parallelepiped.

15. A triangular prism.

16. A triangular pyramid.

17. Prove Theorem I of § 177.

18. Prove Theorem II of § 177.

19. Prove Theorem III of § 177.

20. Prove analytically that if a surface is symmetric with respect to each of two perpendicular planes, it is symmetric with respect to their line of intersection. Is the converse true?

---

* Provided a is positive.

21. Prove the theorem of Ex. 20 geometrically.

22. Prove both analytically and geometrically that if a surface is symmetric with respect to three mutually perpendicular planes, it is symmetric with respect to their point of intersection.

23. Prove both analytically and geometrically that if a surface is symmetric with respect to a perpendicular line and plane, it is symmetric with respect to their point of intersection.

24. What are the conditions that the plane $Ax + By + Cz + D = 0$ be symmetric with respect to (*a*) the xy-plane? (*b*) The x-axis? (*c*) The origin?

Discuss each of the following surfaces according to the method of § 178, and make a careful sketch.

25. $x^2 + y^2 + z^2 = 25.$

26. $z^2 = 4x^2 + 4y^2.$

27. $x^2 + y^2 + z = 4.$

28. $4x^2 - y^2 + 4z^2 = 4a^2.$

29. $y^2 = 4ax.$

30. $x^2 + y^2 = a^2.$

31. $\frac{x^2}{25} + \frac{y^2}{4} - \frac{z^2}{1} = 1.$

32. $\frac{x^2}{25} + \frac{y^2}{4} + \frac{z^2}{1} = 1.$

33. $x^2 + y^2 = 2ay.$

34. $x + y + z^2 = 4.$

35. $x + z - y^2 = 1.$

36. $zx + y = 1.$

37. $2y^2 + 8z^2 + x = 8.$

38. $y^2 + z^2 = 4x.$

39. $xy = 1.$

40. $xy = z.$

179. Generation of a surface by a moving curve. If a curve moves in space in any way, it evidently sweeps out, or generates, a surface of some kind. While a surface may be defined as the locus of a moving point (§ 151), we more often think of it as generated by a curve moving according to a given law. Many examples of this mode of definition will appear in the present chapter. In each problem, unless the contrary is indicated, the generating curve will be understood to keep the same size and shape throughout the motion.

180. Surfaces of revolution. A *surface of revolution* is a surface that can be generated by rotating a curve about a straight line. The straight line is called the *axis of revolution*, and the revolved curve is the *generating curve*, or *generator*. The sections by planes through the axis are *meridian* *sections*, those by planes perpendicular to the axis are *right sections*, or *parallels*. Evidently the meridian sections form a family of equal curves, while the right sections are circles with centers in the axis, and of different radii.

Any curve that lies wholly in the surface and intersects all the right sections may be used to generate the surface, but usually a meridian section is chosen for this purpose.

The more important surfaces of revolution may be listed as follows.

Surface	*Generated by the rotation of*
The *sphere*	A circle about a diameter.
The *prolate spheroid**	An ellipse about its major axis.
The *oblate spheroid**	An ellipse about its minor axis.
The *hyperboloid of revolution of one sheet*	A hyperbola about its conjugate axis.
The *hyperboloid of revolution of two sheets*	A hyperbola about its transverse axis.
The *paraboloid of revolution*	A parabola about its axis.
The *circular cylinder*	A straight line about a line parallel to it.
The *circular cone*	A straight line about a line intersecting it.
The *torus*	A circle about a line in its plane but not intersecting it.

The equations of all of these surfaces, with the exception of the torus, are of the second degree.

In the simpler cases, the equation of a surface of revolution is easily derived when the axis and generating curve are given. The method is best explained by means of

Examples: (*a*) Derive the equation of the cone generated by revolving the line

$$(1) \qquad x + y = 1, z = 0$$

about Ox.

---

* The spheroids are also called *ellipsoids of revolution.*

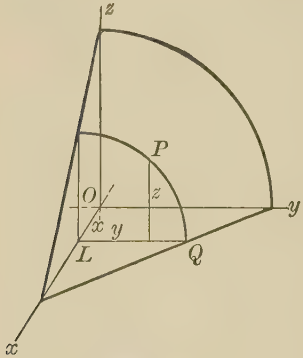


FIG. 124

In the rotation, every point of the generating line describes a circle parallel to the yz-plane, with its center in the x-axis. Let Q be any point of the generating line, and P any point of the circle described by Q, so that P is a general point on the surface. If the coördinates of P be denoted by (x, y, z), those of Q are $(x, y_Q, 0)$, where y_Q denotes the y-coördinate of Q, viz. the distance LQ. We have to obtain an equation involving x, y, and z but not y_Q. (The student should review § 105 at this point.)

The equation of the circular path of Q in its own plane is

$$y^2 + z^2 = y_Q^2. \tag{2}$$

But Q lies on the generating line (1), so that

$$x + y_Q = 1,$$

or

$$y_Q = 1 - x. \tag{3}$$

Eliminating y_Q between (2) and (3), we get

$$y^2 + z^2 = (1 - x)^2,$$

which is the equation of the cone. As a check we note that the traces are, in the xy-plane, the given line

$$y = 1 - x$$

and the line

$$y = -(1 - x),$$

in the zx-plane, the lines

$$z = \pm(1 - x),$$

and in the yz-plane the circle

$$y^2 + z^2 = 1,$$

all of which are seen by the figure to be correct.

(*b*) Derive the equation of the oblate spheroid.

We will generate the spheroid by revolving the ellipse

$$\frac{y^2}{a^2}+\frac{z^2}{b^2}=1,\ x=0$$

about its minor axis Oz. Let $P: (x, y, z)$ be any point of the surface, and $Q: (0, y_Q, z)$ the corresponding point of the generating ellipse. The equation of the circle described by Q is

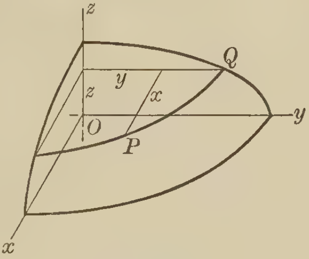


FIG. 125

$$(1)\qquad x^2+y^2=y_Q^2.$$

But we have, since Q is on the ellipse,

$$(2)\qquad \frac{y_Q^2}{a^2}+\frac{z^2}{b^2}=1.$$

Substituting the value of y_Q^2 from (1) in (2), we get

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1.$$

This result may be checked as in the previous example.

EXERCISES

Derive the equations of the surfaces generated as indicated, and in each case sketch the surface. Check by means of the traces.

1. By revolving the line $y = 3x$, $z = 0$ (*a*) about Ox; (*b*) about Oy.
Ans. (*a*) $9x^2 - y^2 - z^2 = 0$.

2. By revolving the line $y + 2z = 4$, $x = 0$ (*a*) about Oz; (*b*) about Oy.
Ans. (*a*) $x^2 + y^2 = 4(z - 2)^2$.

3. By revolving the parabola $y^2 = x$, $z = 0$ (*a*) about its axis; (*b*) about the tangent at the vertex.

4. By revolving the line $z = x - 2$, $y = 0$ (*a*) about Ox; (*b*) about Oz.

5. By revolving the circle $x^2 + z^2 = 1$, $y = 0$ about Oz.

6. By revolving the line $y = 4z + 3$, $x = 0$ (*a*) about Oy; (*b*) about Oz.

7. By revolving the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$, $z = 0$ about each of its axes in turn.

8. By revolving the hyperbola $x^2 - z^2 = a^2$, $y = 0$ about each of its axes in turn.

9. By revolving the line $y = 4$, $z = 0$ about Ox.

10. By revolving the line $x = 2$, $y = 0$ about Oz.

11. By revolving the hyperbola $yz = 1$, $x = 0$ about one of its asymptotes.

12. By revolving the curve $z = y^3$, $x = 0$ (*a*) about Oy; (*b*) about Oz.

13. By revolving the curve $y = \frac{1}{x^2}$, $z = 0$ (*a*) about Ox; (*b*) about Oy.

14. By revolving the line $x = 1$, $y = 2$ about Oz.

15. By revolving the line $y = 2$, $z = 3$ about Ox.

16. By revolving the line $x = -2$, $z = 2$ about Oy.

17. Derive the equation of the prolate spheroid.

18. Derive the equation of the hyperboloid of revolution of one sheet.

19. Derive the equation of the hyperboloid of revolution of two sheets.

20. Derive the equation of the paraboloid of revolution.

Derive the equations of the surfaces generated as indicated, and draw the figures. In each case, tell what kind of surface is obtained.

21. By revolving the line $z = y + 4$, $x = 4$ about the line $x = 4$, $z = 0$. *Ans.* $(x - 4)^2 + z^2 - (y + 4)^2 = 0$.

22. By revolving the circle $(x - 4)^2 + z^2 = 4$, $y = 2$ about the line $x = 4$, $y = 2$. *Ans.* $(x - 4)^2 + (y - 2)^2 + z^2 = 4$.

23. By revolving the parabola $z^2 = x - 1$, $y = 2$ about its axis.

24. By revolving the hyperbola $(y - 2)^2 - (z - 1)^2 = 1$, $x = 3$ about each of its axes in turn.

25. By revolving the line $x = 2$, $y = 0$ about the line $x = 1$, $y = 5$.

26. By revolving the line $z = 4$, $y = 3$ about the line $z = 1$, $y = 2$.

27. By revolving the line $y = x$, $y = 2z$ (*a*) about Ox; (*b*) about Oy; (*c*) about Oz. *Ans.* (*a*) $4y^2 + 4z^2 = 5x^2$.

28. By revolving the line through (0, 0, 0) and (1, 2, 3) about each of the axes in turn. *Ans.* About Oy, $2x^2 + 2z^2 = 5y^2$.

29. By revolving the circle $x^2 + (z - 2)^2 = 1$, $y = 0$ about Ox.
Ans. $(x^2 + y^2 + z^2 - 5)^2 = 16\,(1 - x^2)$.

30. By revolving the circle $(x - b)^2 + y^2 = a^2$, $z = 0$ about Oy.
Ans. $(x^2 + y^2 + z^2 - a^2 - b^2)^2 = 4\,b^2\,(a^2 - y^2)$.

31. Revolve a circle about one of its tangents.

32. What is represented by the equation (cylindrical coördinates) $r = z$? Draw the surface.

33. What is represented by an equation in the cylindrical coördinates r and z, θ being missing?

181. The sphere. A *sphere* is the locus of a moving point that remains at a constant distance from a fixed point. The constant distance is the *radius* and the fixed point the *center*. A line segment through the center terminated by points of the sphere is called a *diameter;* the same term may denote the line in which this segment lies.

The equation of a sphere of given center and radius can be written down at once by the formula for the distance between two points. The equation of a sphere of radius a is, if the center is the origin,

$$x^2 + y^2 + z^2 = a^2; \tag{1}$$

if the center is the point (h, i, k),

$$(x - h)^2 + (y - i)^2 + (z - k)^2 = a^2. \tag{2}$$

Every equation of the form

$$Ax^2 + Ay^2 + Az^2 + Gx + Hy + Iz + K = 0 \tag{3}$$

can be reduced to the form (2) by completing the squares in x, y, and z; hence *every equation of the form* (3) *represents a sphere.* As limiting forms (analogous to those arising in the case of the circle) we have the point-sphere and the imaginary sphere.

182. Tangent plane; tangent lines. If P is any point of a sphere, the plane through P perpendicular to the radius drawn to that point is called the *tangent plane* at P. Any line lying in the tangent plane and passing through the point of contact P is said to be a *tangent line* at P. Thus a sphere has at any point a single tangent plane and infinitely many tangent lines.

EXERCISES

1. Write the equation of the sphere of radius 1 with center at $(1, 0, 0)$. Draw the sphere.

2. Write the equation of the sphere of radius 2 with center at $(1, 2, 2)$. Draw the sphere.

Find the center and radius of each of the following spheres.

3. $x^2 + y^2 + z^2 + 2x - 4z + 1 = 0$.

4. $3x^2 + 3y^2 + 3z^2 + x - 2y + z = 3$.

5. $2x^2 + 2y^2 + 2z^2 = z$.

6. $x^2 + y^2 + z^2 - x - 3y + 4z = 0$.

7. $x^2 + y^2 + z^2 + 4x - 2z + 5 = 0$.

8. Write the equation of a sphere with center at O tangent to the plane $3x - y + 2z + 4 = 0$.

9. Write the equation of a sphere with center at $(2, 1, -3)$ tangent to the plane $x + y - z = 2$.

10. A sphere of radius 2 touches the planes $x - 2y - 2z = 3$, $x + 2y + 2z + 1 = 0$. Find the locus of its center.

11. Find the equation of a sphere with radius $\sqrt{30}$, passing through $(0, 0, 0)$, and having its center on the line $2x + y - z = 0$, $x + 4y - z = 1$. *Ans.* $(x - 2)^2 + (y - 1)^2 + (z - 5)^2 = 30$;
$(x + \frac{134}{59})^2 + (y + \frac{25}{59})^2 + (z + \frac{293}{59})^2 = 30$.

12. Find the equation of a sphere with center $(0, 2, 0)$ whose yz-trace is $(y - 2)^2 + z^2 = 4$.

13. Write the equation in cylindrical coördinates of a sphere of radius a with center at O.

14. Find the equation of a sphere having its center in the x-axis and passing through the points $(0, 2, 0)$, $(2, 0, 4)$.

15. Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 10$ at $(3, 0, -1)$.

16. Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at $(1, 1, -2)$. *Ans.* $x + y - 2z = 6$.

17. Prove in two ways that the line $\frac{x-1}{1} = \frac{y-1}{3} = \frac{z+2}{2}$ touches the sphere $x^2 + y^2 + z^2 = 6$ at $(1, 1, -2)$. (Cf. Ex. 16.)

18. Find the equations of the line parallel to the xy-plane, touching the sphere $x^2 + y^2 + z^2 = 6z$ at $(2, 1, 1)$. *Ans.* $2x + y = 5$, $z = 1$.

19. Find the tangent plane at $(1, -1, 1)$ to the sphere $2x^2 + 2y^2 + 2z^2 - 3x - y + z - 5 = 0$. *Ans.* $x - 5y + 5z = 11$.

20. Show that a line can be drawn through $(1, 0, 2)$ touching the sphere of Ex. 19 at $(1, -1, 1)$, and find the equations of that line.

21. Write the equations of the line parallel to the xy-plane touching the sphere of Ex. 19 at $(1, -1, 1)$.

Ans. $\dfrac{x-1}{5} = \dfrac{y+1}{1} = \dfrac{z-1}{0}$.

22. Find the equation of a plane parallel to the plane $x + y + z = 0$ and touching the sphere $x^2 + y^2 + z^2 - 2x - y - 5z = 2$.

Ans. $x + y + z = 4 \pm \frac{1}{2}\sqrt{114}$.

23. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = a^2$ at (x_1, y_1, z_1). *Ans.* $x_1x + y_1y + z_1z = a^2$.

24. Prove that a sphere is determined by four points.

25. Find the equation of the sphere through the points $(0, 0, 0)$, $(-1, 0, 0)$, $(0, 4, 0)$, $(1, 2, -1)$.

Ans. $x^2 + y^2 + z^2 + x - 4y - z = 0$.

26. A sphere of radius $\sqrt{6}$ touches the plane $2x + y + z = 5$ at $(1, 1, 2)$. Find its equation. *Ans.* $(x-3)^2 + (y-2)^2 + (z-3)^2 = 6$;
$(x+1)^2 + y^2 + (z-1)^2 = 6$.

27. Find the equation of a sphere of radius 3 touching the plane $4x + y + 8z = 9$ at $(1, 1, \frac{1}{2})$.

28. Find the equation of a sphere touching the plane $3x + y + z = 6$ at $(1, 0, 3)$ and passing through $(2, 2, 1)$.

Ans. $(x - \frac{11}{2})^2 + (y - \frac{3}{2})^2 + (z - \frac{9}{2})^2 = \frac{99}{4}$.

29. Find the equation of a sphere touching the plane $x + y - z = 4$ at $(1, 2, -1)$ and passing through $(5, 2, 3)$.

30. Find the equation of the sphere touching the plane $x + y + 2z = 4$ at $(2, 0, 1)$ and passing through $(0, 1, 0)$.

Ans. $(x-1)^2 + (y+1)^2 + (z+1)^2 = 6$.

183. Quadric surfaces. A surface whose equation is of the second degree is called a *quadric surface*. The quadrics occupy much the same place among surfaces that the conics hold among curves: next to the plane, they are by far the most important class of surfaces.

The quadrics are of nine species: the ellipsoid (of which the sphere is a special case), the hyperboloids (two species), the paraboloids (two species), the quadric cylinder (three species), and the quadric cone. These surfaces will now be taken up in order.

184. The ellipsoid. Let there be given two fixed ellipses lying in perpendicular planes and having one axis in common (the curves AQB, ARC of Fig. 126). If a variable ellipse QPR, lying in a plane perpendicular to the planes of the fixed ellipses, be imagined to move so that the ends of its axes always lie in the fixed ellipses, this moving ellipse generates a surface called an *ellipsoid*. The fixed ellipses are the *directing curves*, the variable ellipse is the *generating curve*.

To obtain the equation of the ellipsoid, let the directing curves be the ellipses (Fig. 126)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0, \tag{1}$$

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1,\ y=0, \tag{2}$$

and let P: (x, y, z) be any point of the surface, Q and R the corresponding points on the directing curves. Since P, Q, and R all have the same x-coördinate, the coördinates of Q and R may be denoted by $(x, y_Q, 0)$ and $(x, 0, z_R)$ respectively. Now the equation of the ellipse QPR referred to LQ, LR as axes is evidently

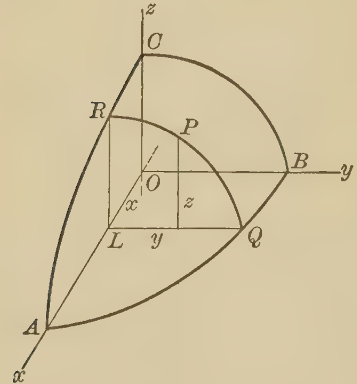


Fig. 126

$$\frac{y^2}{{y_Q}^2}+\frac{z^2}{{z_R}^2}=1; \tag{3}$$

further, since Q and R lie on the ellipses (1) and (2) respectively, we have

$$\frac{x^2}{a^2}+\frac{{y_Q}^2}{b^2}=1, \tag{4}$$

(5) $$\frac{x^2}{a^2} + \frac{z_R^2}{c^2} = 1.$$

Substituting the values of y_Q^2 and z_R^2 from (4) and (5) in (3), we get

$$\frac{y^2}{b^2\left(1 - \frac{x^2}{a^2}\right)} + \frac{z^2}{c^2\left(1 - \frac{x^2}{a^2}\right)} = 1,$$

or

(4) $$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.}$$

As a check we note that the *xy*- and *zx*-traces are the given directing curves.

From equation (**4**), or directly from the definition, we deduce at once the following properties of the surface:

(*a*) The ellipsoid has three planes of symmetry, called the *principal planes*, three lines of symmetry, and a center. The lines of symmetry, or the segments of those lines included within the surface, are the *axes* of the ellipsoid. The axes are of lengths 2 a, 2 b, 2 c.

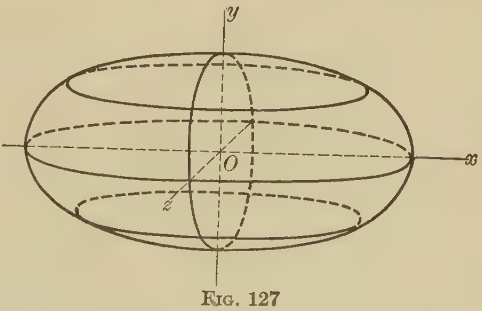


FIG. 127

(*b*) The sections by the planes $x = k$ are, if* $|k| < a$, similar † ellipses; if $k = \pm a$, a point; if $|k| > a$, imaginary. Likewise the sections $y = k$ are similar ellipses within the interval from $-b$ to b; the sections $z = k$ are similar ellipses from $-c$ to c. The ellipses decrease in size as $|k|$ increases.

---

* The symbol $|k|$ is used to denote the absolute or numerical value of k apart from sign. For example, $|-3| = 3$.

† See Ex. 17, p. 174.

(*c*) The surface is a closed surface lying within the parallelepiped bounded by the six planes $x = \pm a$, $y = \pm b$, $z = \pm c$.

Evidently any two of the three sections by the principal planes may be used as directing curves.

Every equation of the form

$$Ax^2 + By^2 + Cz^2 + K = 0,$$

where A, B, C are positive and K is negative, represents an ellipsoid referred to its principal planes as coördinate planes. As limiting forms we have, when $K = 0$, the *point ellipsoid*, when K is positive, the *imaginary ellipsoid*.

When two of the semi-axes a, b, c are equal, so that the sections by planes perpendicular to the third axis are circles, the surface becomes an ellipsoid of revolution: if the equal axes are shorter than the third, a prolate spheroid, if longer, an oblate spheroid.

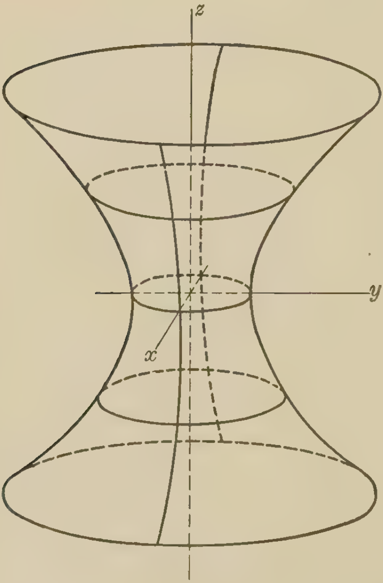


FIG. 128

185. The hyperboloid of one sheet. If the generating ellipse follows as directing curves two hyperbolas lying in perpendicular planes and having a common conjugate axis, the surface formed is called a *hyperboloid of one sheet*.

The equation of the hyperboloid of one sheet can be derived by the method used for the ellipsoid. If the directing curves are the hyperbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \; y = 0$$

and

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,\ x = 0,$$

as in Fig. 128, the equation of the surface is found to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{5}$$

From this we deduce the following properties:

(*a*) The surface has three planes of symmetry, or *principal planes*, three lines of symmetry, or *axes*, and a center.

(*b*) Sections by planes parallel to either of the directing curves are hyperbolas. Those by planes $x = k$ have the transverse axis parallel to Oy if $|k| < a$, parallel to Oz if $|k| > a$; those by planes $y = k$ have the transverse axis parallel to Ox if $|k| < a$, parallel to Oz if $|k| > a$. The sections by the planes $x = \pm a$ and $y = \pm b$ are intersecting lines.

(*c*) The sections by planes $z = k$ are similar ellipses, increasing in size as $|k|$ increases, the smallest being cut by the plane $z = 0$.

(*d*) The surface is a connected, open surface extending to infinity in both directions along the z-axis.

If the directing curves have equal transverse axes, so that $a = b$, the elliptic sections become circular and the surface is the hyperboloid of revolution of one sheet.

Instead of deriving equation (5) from the definition, we will apply the method to a numerical

Example: Derive the equation of the hyperboloid whose directing curves are

(1) $9x^2 - y^2 = 9,\ z = 0$

and

(2) $z^2 - y^2 = 9,\ x = 0.$

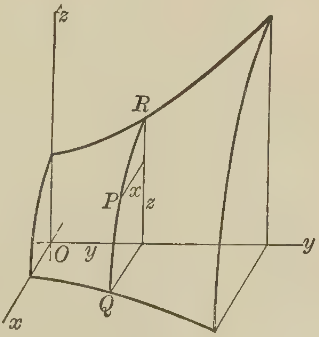


Fig. 129

Let $P: (x, y, z)$ be any point of the surface, Q and R the corresponding points on the directing hyperbolas. Then Q and R have the coördinates $(x_Q, y, 0)$ and $(0, y, z_R)$ respectively. The equation of the generating ellipse QPR in its plane is

$$\frac{x^2}{x_Q{}^2} + \frac{z^2}{z_R{}^2} = 1. \tag{3}$$

But, since Q and R lie on the hyperbolas (1) and (2), we have

$$9\, x_Q{}^2 - y^2 = 9, \tag{4}$$

$$z_R{}^2 - y^2 = 9. \tag{5}$$

Eliminating the parameters x_Q, z_R from (3), (4), and (5), we find

$$9\, x^2 + z^2 - y^2 = 9.$$

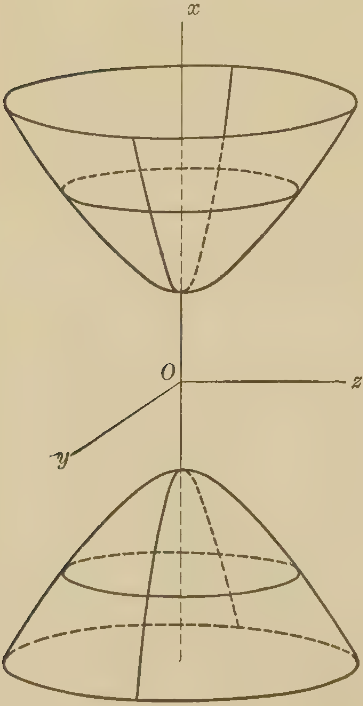


Fig. 130

The answer may be checked by finding the traces of the surface.

186. The hyperboloid of two sheets. The *hyperboloid of two sheets* is the surface generated by a variable ellipse that follows as directing curves two fixed hyperbolas lying in perpendicular planes and having a common transverse axis. If the directing curves lie in the zx- and xy-planes, the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{6}$$

From equation (6) we deduce the following:

(*a*) There are three planes and axes of symmetry, and a center.

(*b*) Sections by planes $y = k$, and by planes $z = k$, are similar hyperbolas with transverse axis parallel to Ox.

(*c*) Sections by planes $x = k$ are, if $|k| < a$, imaginary; if $k = \pm a$, a point; if $|k| > a$, similar ellipses, increasing in size as $|k|$ increases.

(*d*) The surface consists of two disconnected sheets, one in the region $x > a$, the other in the region $x < -a$, each opening out larger as x increases numerically.

If the directing hyperbolas are equal, so that $b = c$, the elliptic sections become circles, and the surface is the hyperboloid of revolution of two sheets.

EXERCISES

Derive the equations of the quadric surfaces generated by a variable ellipse following the given conics as directing curves. In each case draw the figure, classify the surface, and determine whether it is a surface of revolution. Check the answers.

1. $\frac{x^2}{4} + \frac{y^2}{9} = 1, z = 0$ and $\frac{x^2}{4} + \frac{z^2}{1} = 1, y = 0.$

2. $\frac{x^2}{16} + \frac{z^2}{64} = 1, y = 0$ and $\frac{y^2}{1} + \frac{z^2}{64} = 1, x = 0.$

3. $\frac{x^2}{9} - \frac{y^2}{9} = 1, z = 0$ and $\frac{z^2}{2} - \frac{y^2}{9} = 1, x = 0.$

4. $\frac{y^2}{4} - \frac{x^2}{1} = 1, z = 0$ and $\frac{y^2}{4} - \frac{z^2}{3} = 1, x = 0.$

5. $4z^2 - y^2 = 1, x = 0$ and $4z^2 - x^2 = 1, y = 0.$

6. $y^2 - x^2 = 2, z = 0$ and $4z^2 - 3x^2 = 6, y = 0.$

7. $3x^2 + 4z^2 = 12, y = 0$ and $y^2 + z^2 = 3, x = 0.$

8. $x^2 - 4z^2 + 1 = 0, y = 0$ and $9y^2 - 16z^2 + 4 = 0, x = 0.$

9. $2x^2 + 5y^2 = 10, z = 0$ and $5y^2 + 2z^2 = 10, x = 0.$

10. $z^2 - x^2 = 1, y = 0$ and $y^2 - 2x^2 = 2, z = 0.$

11. $\frac{(x-1)^2}{1} + \frac{(y-2)^2}{4} = 1, z = 1$ and $\frac{(y-2)^2}{4} + \frac{(z-1)^2}{3} = 1,$ $x = 1.$

12. $\frac{(y-2)^2}{4} - \frac{x^2}{1} = 1, z = 3$ and $\frac{(y-2)^2}{4} - \frac{(z-3)^2}{1} = 1, x = 0.$

13. $(x-4)^2-(y-4)^2=1$, $z=3$ and $(z-3)^2-(y-4)^2=1$ $x=4$.

Classify, describe, and sketch the following surfaces.

14. $4x^2+4y^2+4z^2=1$.

15. $x^2-y^2-z^2=1$.

16. $\frac{z^2}{4}+\frac{y^2}{2}+\frac{x^2}{1}=1$.

17. $\frac{z^2}{3}+\frac{x^2}{4}-\frac{y^2}{4}=1$.

18. $x^2+2y^2-3z^2+6=0$.

19. $x^2+2y^2+z^2=0$.

20. $2x^2-y^2-z^2+4=0$.

21. $y^2-4x^2-9z^2=1$.

22. $3x^2+2y^2+6z^2+6=0$.

23. $z^2-4x^2-y^2+4=0$.

24. $x^2+4y^2+z^2=16$.

25. $4x^2+4y^2+z^2=16$.

26. Derive equation (5), § 185.

27. Derive equation (6), § 186.

28. Show that the equation $Ax^2+By^2+Cz^2+K=0$, where A, B, C, and K are all different from 0, represents an ellipsoid or hyperboloid whose axes lie in the coördinate axes. How many points are required to determine this surface?

In Exs. 29–35, the equation of the surface is assumed to be of the form $Ax^2+By^2+Cz^2+K=0$.

29. Find the equation of a quadric through the points (0, 0, 2), (1, 1, $\sqrt{5}$), ($2\sqrt{5}$, 2, 4). *Ans.* $2z^2-x^2-y^2=8$.

30. Find the equation of a quadric through the points (2, 1, 1), (3, 0, 0), (1, 1, 2). *Ans.* $x^2+4y^2+z^2=9$.

31. Find the equation of a quadric through the points (7, 1, 8), (1, 5, 4), (8, 2, 10). *Ans.* $2x^2+2y^2-z^2=36$.

32. Find the equation of a quadric through the points (0, 1, 3), (3, 5, 2), (4, 2, 1). *Ans.* $177x^2+8y^2+357z^2=3221$.

33. Find the equation of a quadric surface of revolution about the z-axis, through (2, 1, 1) and (0, 2, 2). *Ans.* $3x^2+3y^2+z^2=16$.

34. Find the equation of a quadric surface of revolution about the y-axis, through (3, 4, 2) and (4, 1, 3). *Ans.* $5x^2+4y^2+5z^2=129$.

35. Find the equation of a quadric surface of revolution about Ox, through (1, 1, 3) and (3, 2, 4). *Ans.* $5x^2-4y^2-4z^2+35=0$.

36. The hyperboloid of revolution of one sheet may be defined as the surface generated by revolving one of two skew lines about the other. Prove this statement by revolving the line $y=h$, $z=mx$ about Ox. Is the proof general? *Ans.* $y^2+z^2-m^2x^2=h^2$.

37. Find the equation of the hyperboloid generated by revolving the line $y = x$, $z = 2$ about Ox. *Ans.* $y^2 + z^2 - x^2 = 4$.

38. Find the equation of the hyperboloid generated by revolving the line $y = 3$, $z = 2x$ about Oz.

39. Find the equation of the hyperboloid generated by revolving the line $x = 4$, $3y - 2z = 0$ about Oy.

40. A point moves so that the sum of its distances from two fixed points is constant. Show that its locus is a prolate spheroid. (Take the fixed points as $(\pm c, 0, 0)$.)

41. A point moves so that the difference of its distances from two fixed points is constant. Show that its locus is a hyperboloid of revolution of two sheets.

42. A point moves so that its distance from a fixed point is in a constant ratio e to its distance from a fixed plane. Show that if $e < 1$, the locus is a prolate spheroid; if $e > 1$, a hyperboloid of revolution of two sheets. (Take the fixed point as $(ae, 0, 0)$, the fixed plane as $x = \frac{a}{e}\cdot$)

187. The elliptic paraboloid. Let there be given two fixed parabolas lying in perpendicular planes, having a common vertex and axis, and opening in the *same* direction. The surface generated by a variable ellipse following these parabolas as directing curves is called an *elliptic paraboloid.* When the directing curves are given, the equation of the surface may be found in the usual way.

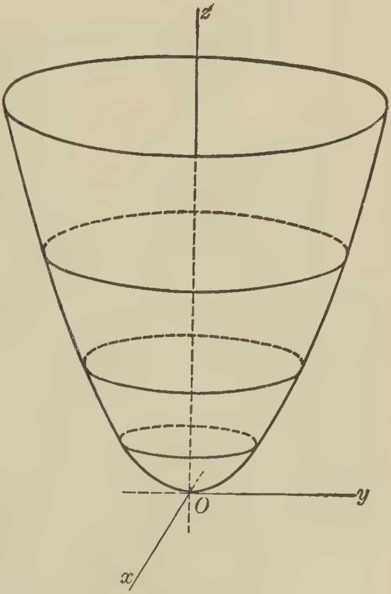


FIG. 131

The same surface may also be generated by moving one of the given parabolas parallel to its original position in such a way that its vertex lies always on the other parabola, the moving parabola keeping its original size throughout.

If the directing parabolas are

$$\frac{y^2}{b^2} = \frac{z}{c},\ x = 0,$$

$$\frac{x^2}{a^2} = \frac{z}{c},\ y = 0,$$

the equation of the paraboloid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}. \tag{7}$$

From equation (7) we obtain the following results:

(*a*) The elliptic paraboloid has two planes of symmetry; it also has one line of symmetry, called the *axis* of the surface. The axis intersects the surface in a single point, called the *vertex*.

(*b*) The sections by planes $x = k$, and those by planes $y = k$, are equal parabolas with axis parallel to Oz, and opening upward or downward according as c is positive or negative.

(*c*) Sections by planes $z = k$ are, when k has the same sign as c, similar ellipses increasing in size as k increases; when $k = 0$, a point; when k and c have opposite signs, imaginary.

(*d*) The surface lies entirely on one side of the xy-plane, and extends to infinity along Oz, opening upward or downward according as c is positive or negative.

When $a = b$, the surface is the *paraboloid* of *revolution*.

188. The hyperbolic paraboloid. Let there be given two parabolas lying in perpendicular planes, having a common vertex and axis, and opening in *opposite* directions. If one of these parabolas, keeping the same size throughout, be moved parallel to its original position in such a way that its vertex always lies on the other parabola, the surface generated is called a *hyperbolic paraboloid*.

If the given parabolas are

$$\frac{x^2}{a^2} = \frac{z}{c}, \; y = 0$$

and

$$\frac{y^2}{b^2} = -\frac{z}{c}, \; x = 0,$$

the equation of the paraboloid is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}. \tag{8}$$

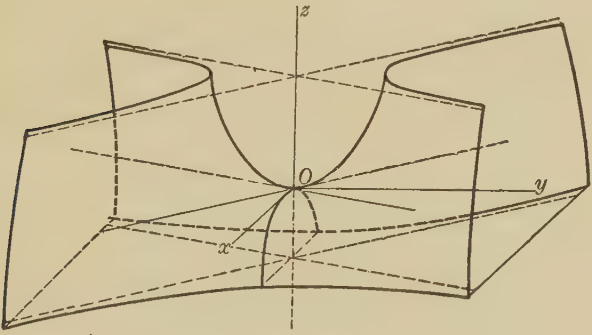


FIG. 132

From (8) we draw the following conclusions:

(*a*) The surface has two planes of symmetry, an *axis* and a *vertex*.

(*b*) Sections by planes $x = k$ are equal parabolas opening in the negative or positive direction of z according as c is positive or negative; those by planes $y = k$ are equal parabolas opening in the positive or negative direction of z according as c is positive or negative.

(*c*) Sections by planes $z = k$ are hyperbolas having the transverse axis parallel to Ox when k has the same sign as c, parallel to Oy when k and c have opposite signs. The trace in the xy-plane is two intersecting lines.

(*d*) The surface is "saddle-shaped," as shown in Fig. 132.

The hyperbolic paraboloid cannot in any case be a surface of revolution.

Instead of deriving equation (8), we will solve the following numerical

Example: Find the equation of the hyperbolic paraboloid generated by moving the parabola

$$z^2 = -y,\ x = 0 \tag{1}$$

along the parabola

$$x^2 = 4y,\ z = 0. \tag{2}$$

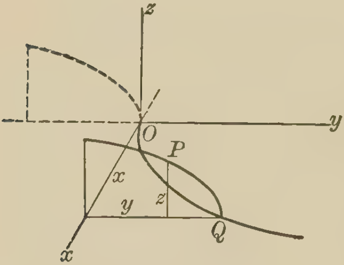


FIG. 133

Let $P: (x, y, z)$ be any point of the surface, and $Q: (x, y_Q, 0)$ the corresponding point on the fixed parabola. The equation of the moving parabola (1) in the position QP is

$$z^2 = -(y - y_Q). \tag{3}$$

Since Q lies on the parabola (2) we have

$$x^2 = 4y_Q. \tag{4}$$

Multiplying (3) by 4 and subtracting from (4), we get

$$x^2 - 4z^2 = 4y.$$

As a check, we note that the yz- and xy-traces are the given parabolas.

EXERCISES

In each of the following cases, derive the equation of the paraboloid generated by moving one of the given parabolas along the other, first drawing a figure.

1. $x^2 = z,\ y = 0$ and $y^2 = 3z,\ x = 0$.

2. $3x^2 = 4y,\ z = 0$ and $z^2 = y,\ x = 0$.

3. $2z^2 = 3x,\ y = 0$ and $5y^2 = 8x,\ z = 0$.

4. $y^2 = 4ax,\ z = 0$ and $z^2 = 4ax,\ y = 0$.

5. $z^2 = 4x,\ y = 0$ and $y^2 = -2x,\ z = 0$.

6. $x^2 - 4y = 0,\ z = 0$ and $z^2 + 4y = 0,\ x = 0$.

7. $5x^2 + 4z = 0,\ y = 0$ and $3y^2 - 2z = 0,\ x = 0$.

8. $x^2 + 2y = 0,\ z = 0$ and $4z^2 + y = 0,\ x = 0$.

9. $4y^2 + x = 0,\ z = 0$ and $z^2 - 6x = 0,\ y = 0$.

10. Derive equation (**7**) by the method indicated in the second paragraph of § 187.

11. Derive equation (**7**) by using an ellipse as generating curve.

12. Derive equation (**8**), § 188.

Classify, describe, and sketch the following surfaces.

13. $3x^2 + z^2 - 4y = 0$.

14. $x^2 + y^2 = 1 - 4z^2$.

15. $4x^2 - z^2 + 4y = 0$.

16. $z^2 - 2y^2 - 4x = 0$.

17. $y^2 - 2x^2 - 4 = z^2$.

18. $y^2 + z^2 = 6x$.

19. $4x^2 - 9y^2 = 0$.

20. $2x^2 + 3z^2 + 2y = 0$.

21. $z^2 = 4x^2 + 4y^2 - 4$.

22. $x^2 - y^2 - z = 0$.

23. How many points are required to determine a paraboloid whose vertex and axis are given?

24. Find the equation of a paraboloid with vertex at O, axis Oy, and passing through $(1, -2, 1)$, $(-3, -3, 2)$. *Ans.* $x^2 - 3z^2 = y$.

25. Find the equation of a paraboloid with vertex at O, axis Ox, and passing through $(3, 1, 2)$, $(3, 3, 0)$.

26. Find the equation of a paraboloid with vertex at O, axis Oz, and passing through $(3, 4, 2)$, $(5, 1, 1)$.

27. Find the equation of a paraboloid of revolution with vertex at O, axis Oy, and passing through $(6, 4, 3)$.

28. For the surface shown in Fig. 132, is the constant c positive or negative?

189. Ruled surfaces. A surface that can be generated by a moving straight line is called a *ruled surface*. It appears from the definition that such a surface is completely covered by straight lines: these lines are called *generators*.

Of the quadrics, the hyperboloid of one sheet, the hyperbolic paraboloid, the three cylinders, and the cone are ruled surfaces, as will appear in the next few articles.

190. The hyperboloid of one sheet as a ruled surface. Let us write the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

in the form

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{x^2}{a^2},$$

and this in turn in the form

$$(1)\qquad \left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{y}{b}-\frac{z}{c}\right)=\left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right).$$

Consider now the family of straight lines represented by the two equations

$$(2)\qquad k\left(\frac{y}{b}+\frac{z}{c}\right)=1+\frac{x}{a},\quad \frac{y}{b}-\frac{z}{c}=k\left(1-\frac{x}{a}\right),$$

where k is an arbitrary constant. Upon multiplying both members of (1) by k, it is easily seen that that equation is satisfied whenever equations (2) are both satisfied, so that every point on every one of the lines (2) lies on the hyperboloid — i.e. the hyperboloid contains the entire family of lines.

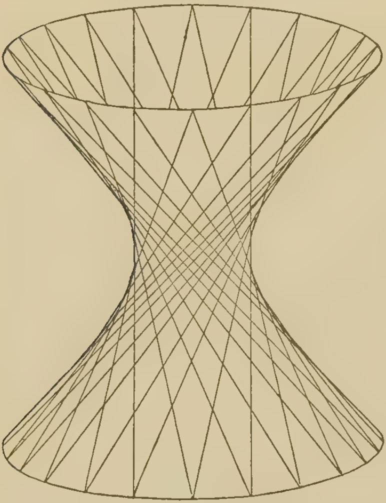


FIG. 134

It remains to prove, conversely, that these lines entirely cover the surface — i.e. that every point of the hyperboloid lies on one of the lines. To show this, we note that through every point of space, and hence through every point of the hyperboloid, there passes one member of the family of planes

$$(3)\qquad k\left(\frac{y}{b}+\frac{z}{c}\right)=1+\frac{x}{a},$$

since when the coördinates of any point are substituted in (3) a value of k is determined*. But when equations (1) and (3) are both satisfied, it is evident that the second of equations (2) also holds, which completes the proof.

Likewise the hyperboloid contains, and is completely covered by, the family of lines

$$k'\left(\frac{y}{b}-\frac{z}{c}\right)=1+\frac{x}{a},\quad \frac{y}{b}+\frac{z}{c}=k'\left(1-\frac{x}{a}\right).$$

Thus we have the

THEOREM: *The hyperboloid of one sheet is a ruled surface containing two distinct families of rectilinear generators.*

191. The hyperbolic paraboloid as a ruled surface. Argument similar to that of § 190 shows that the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$$

contains every line of the family

$$(1)\quad k\left(\frac{x}{a}+\frac{y}{b}\right)=\frac{z}{c},\quad \frac{x}{a}-\frac{y}{b}=k,$$

and that every point of the paraboloid lies on one of these lines. Similar statements hold for the family of lines

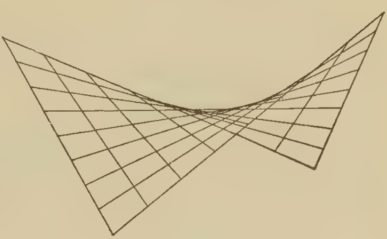


FIG. 135

$$k'\left(\frac{x}{a}-\frac{y}{b}\right)=\frac{z}{c},\quad \frac{x}{a}+\frac{y}{b}=k'.$$

This proves

THEOREM I: *The hyperbolic paraboloid is a ruled surface containing two distinct families of rectilinear generators.*

Every line of the family (1) is parallel to the fixed plane

$$(2)\qquad \frac{x}{a}-\frac{y}{b}=0$$

---

* The apparent difficulty arising in the case of a point in the plane $\frac{y}{b}+\frac{z}{c}=0$ disappears upon replacing k by $\frac{1}{k}$. Cf. § 44.

and intersects the fixed lines

$$(3) \qquad \frac{x}{a} + \frac{y}{b} = 0, \quad z = 0$$

and

$$(4) \qquad K\left(\frac{x}{a} - \frac{y}{b}\right) = \frac{z}{c}, \quad \frac{x}{a} + \frac{y}{b} = K,$$

where, in (4), K may have any definite value. Further, the lines (3) and (4) are skew (i.e. not parallel or intersecting), and each of them intersects the plane (2). (In Exs. 24–28 below, the student is asked to prove each of these statements.) We have thus established an important property of the hyperbolic paraboloid, which may be stated as

THEOREM II: *The hyperbolic paraboloid may be generated by a straight line moving parallel to a fixed plane and intersecting two fixed skew lines which themselves intersect the fixed plane.*

The equation of a paraboloid generated in the manner just described may be found as in the following

Example: Find the equation of the hyperbolic paraboloid generated by a line moving parallel to the xy-plane and intersecting the lines $x + z = a$, $y = 0$ and $y = b$, $x = 0$.

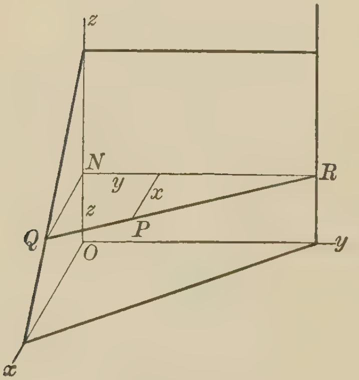


FIG. 136

Let P: (x, y, z) be any point of the surface, and QR (Fig. 136) the generating line through P. The equation of that line referred to NQ, NR as axes is, in the intercept form,

$$\frac{x}{x_Q} + \frac{y}{y_R} = 1.$$

But

$$x_Q + z = a, \; y_R = b,$$

whence

$$\frac{x}{a-z}+\frac{y}{b}=1,$$

or

$$yz - bx - ay - bz + ab = 0.$$

The answer may be checked by finding the traces in the coordinate planes.

EXERCISES

Prove that each of the following equations represents a ruled surface and find the equations of a family of generators. Sketch each surface.

1. $y^2 - x - z = 1.$ *Ans.* $y = k,\ x + z = k^2 - 1.$

2. $x + y + z = 4.$

3. $x + y + z^2 = 4.$

4. $y^2 = 4\,ax.$

5. $x^2 + y^2 = 2\,ay.$

6. $xy = z^2.$

7. $x^2 + y^2 = yz.$

8. $x^2 + y^2 = z^2.$

9. $z = 1 - x^2y.$

10. $xy + yz = 1.$

11. $x^2z - x^2 + y = 0.$

In each of the following cases, find two families of rectilinear generators.

12. $z = xy.$

13. $y = 1 - zx.$

14. $\frac{x^2}{4} - \frac{z^2}{9} = y.$

15. $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1.$

16. $x^2 - z^2 = y^2 - 1.$

17. $z^2 - 4\,y^2 + 2\,x = 0.$

Find the equations of the ruled surfaces having the following families of lines as generators.

18. $k\,(x + 1) = y,\ z = k.$

19. $x - 2\,z = 1 + k,\ y = k.$

20. $x = \sqrt{2\,k + k^2},\ y = k.$

21. $x + y = k\,(1 - 2\,z),\ k\,(x - y) = 1 + 2\,z.$

22. $y - 2\,z = k\,(1 + x),\ k\,(y + 2\,z) = 1 - x.$

23. In the example of § 191, find the other family of generators. Draw the surface.

24. Prove that every line of the family (1), § 191, is parallel to the plane (2).

25. Prove that every line of the family (1), § 191, intersects the line (3). Find the point of intersection.

26. Prove that every line of the family (1), § 191, intersects the line (4), whatever the value of K.

27. Prove that the lines (3) and (4), § 191, are skew.

28. Prove that the lines (3) and (4), § 191, intersect the plane (2).

Find the equations of the following hyperbolic paraboloids, making a sketch in each case.

29. Generated by a line parallel to the xy-plane intersecting the lines $z = x$, $y = 0$ and $y = a$, $x = 0$. *Ans.* $yz + ax - az = 0$.

30. Generated by a line parallel to the zx-plane intersecting the lines $x = 1$, $z = 0$ and $y = 2z$, $x = 0$. *Ans.* $xy + 2z - y = 0$.

31. Generated by a line parallel to the xy-plane intersecting the lines $x = a$, $y = 0$ and $z = x$, $y = b$. *Ans.* $yz - bx - ay + ab = 0$.

32. Generated by a line parallel to the zx-plane intersecting the lines $x = y = z$ and $x = a$, $z = 0$.

33. Generated by a line parallel to the yz-plane intersecting the lines $y = 2$, $z = 1$ and $z = x$, $y = 0$. *Ans.* $xy - 2x - y + 2z = 0$.

34. Generated by a line parallel to the plane $x + y = 0$ intersecting the x-axis and the line $z = a$, $x = 0$. *Ans.* $yz + zx = ay$.

35. Prove that every equation of the second degree containing no square terms and not more than two product terms represents a ruled quadric.

36. Prove that every equation of the second degree in which the second degree terms form a perfect square represents a ruled quadric.

192. Cylinders. A *cylinder* is the surface described by a moving line which remains parallel to its original position and always intersects a fixed curve, called the *directing curve.* Thus the cylinder is the ruled surface all of whose generators are parallel.

The section by any plane perpendicular to the generators is called a *right section;* it is geometrically obvious that all right sections, and in fact all parallel plane sections, are equal curves. Evidently any curve that lies wholly in the surface and intersects all the generators may serve as directing curve, but usually a right section is chosen. If the right section has a center, the line through this center parallel to the generators is the *axis* of the cylinder.

193. Equations in two variables: cylinders perpendicular to a coördinate plane. In the present article we will investigate the locus of an equation in two variables — i.e. an equation in which x, y, or z is missing — beginning with an

Example: Determine the locus of the equation

$$x^2 + y^2 = a^2. \tag{1}$$

By § 144, the locus is a surface of some kind. In the xy-plane, the equation represents a circle. At any point M of this circle erect a perpendicular to the xy-plane, and let P: (x, y, z) be any point of the perpendicular. Now the coördinates of M satisfy (1), and those of P must do the same, since its x- and y-coördinates are the same as those of M, and z does not occur in the equation. Thus P is on the surface, and since P is any point of the perpendicular at M, that entire line must lie in the surface. Further, since M is any point of the circle, the surface includes all lines perpendicular to the xy-plane through points of the circle. On the other hand, if a point does not lie in one of these lines its coördinates obviously cannot satisfy the equation, since its x- and y-coördinates cannot be identical with those of any point of the circle. Thus the equation represents a *cylinder with generators perpendicular to the xy-plane*, the directing curve being the curve represented by the given equation in that plane.

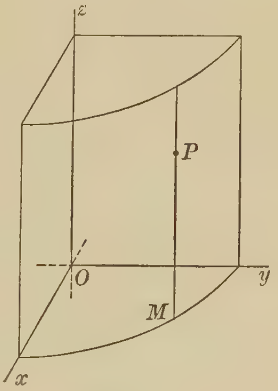


FIG. 137

Evidently a strictly analogous result holds for any equation in two variables. Conversely, it is easily seen that the equation of every cylinder with generators perpendicular to a coördinate plane involves only two variables. Thus we have the important

THEOREM: *Every equation in two variables represents a cylinder whose generators are perpendicular to the plane of the two variables and whose directing curve is the curve represented by the given equation in that plane; and conversely.*

194. Quadric cylinders. Given any cylinder, let one of the coördinate planes, say the xy-plane, be taken perpendicular to the generators. Then, by § 193, the equation of the cylinder is identical in form with that of its xy-trace. It follows that the degree of the equation of a cylinder depends upon the nature of its right section, so that a cylinder may or may not be a quadric surface; in fact, we have the

THEOREM: *A cylinder is a quadric surface if and only if its right section is a conic.*

A quadric cylinder is called *elliptic*, *parabolic*, or *hyperbolic* according as its right section is an ellipse, a parabola, or a hyperbola. The cylinder of revolution, or circular cylinder, is of course a special case of the elliptic cylinder.

195. The equation of a cylinder: general case. The equation of a cylinder can be found if the direction cosines of the generators and the equations of the directing curve are given; sufficient explanation of the method is furnished by the following

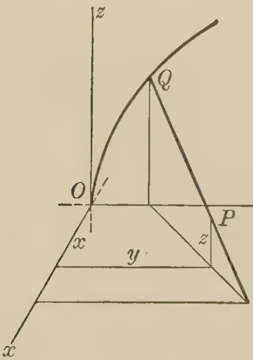


FIG. 138

Example: Find the equation of the cylinder whose directing curve is the parabola

$$z^2 = 4\,y,\ x = 0,$$

and whose generators have direction cosines proportional to 1, 1, -1.

Let $P: (x,\ y,\ z)$ be any point of the surface, and

Q: $(0, y_Q, z_Q)$ the point in which the generator through P intersects the directing curve. The equations of the line QP are

$$\frac{x}{1} = \frac{y - y_Q}{1} = \frac{z - z_Q}{-1},$$

whence

$$y_Q = y - x,\ z_Q = z + x.$$

Further,

$$z_Q^2 = 4\,y_Q.$$

Substituting, we find

$$(z + x)^2 = 4\,(y - x).$$

EXERCISES

Draw the following cylinders.

1. $\frac{x^2}{4} + \frac{y^2}{9} = 1.$

2. $x^2 = 4\,ay.$

3. $z^2 - x^2 = a^2.$

4. $z = 1 - x^2.$

5. $yz = c^2.$

6. $y = z^3.$

Prove that the following equations represent cylinders.

7. $x^2 - y - 2\,z = 0.$

8. $z^2 = x + y + z.$

9. $xy + yz = 1.$

10. $2\,yz + 3\,zx + 4 = 0.$

11. Prove that every equation in which two of the variables occur only in the terms of the first degree represents a cylinder with generators parallel to the plane of those variables.

12. Prove analytically that if the right section of a cylinder has a line of symmetry the cylinder itself is symmetric with respect to that line and also with respect to the plane through that line parallel to the generators.

13. Prove that if a cylinder has an axis it is symmetric with respect to the axis and also with respect to every point of the axis.

14. What is represented by an equation in cylindrical coördinates in which z is missing?

15. What is represented by the equation (cylindrical coördinates) $r = a$?

16. Write the equation in cylindrical coördinates of the cylinder whose right section is the conic $r = \frac{l}{1 + e\cos\theta}$, $z = 0$.

17. Find the equation of the cylinder whose directing curve is the circle $x^2 + y^2 = 1$, $z = 0$ and whose generators are parallel to the radius vector of the point (1, 2, 3). *Ans.* $(3x - z)^2 + (3y - 2z)^2 = 9$.

18. Find the equation of the cylinder whose directing curve is $x^2 - z^2 = a^2$, $y = 0$, and whose generators are parallel to the radius vector of the point (a, a, a). *Ans.* $x^2 - z^2 - 2xy + 2yz = a^2$.

19. Find the cylinder whose directing curve is $y^2 = z$, $x = 0$ and whose generators are perpendicular to the plane $2x - 3y + z = 4$.
Ans. $9x^2 + 12xy + 4y^2 + 2x - 4z = 0$.

20. Find the cylinder whose directing curve is $x^2 + y^2 = a^2$, $z = 0$ and whose generators are parallel to the line $y = z$, $x = 0$. Sketch the surface. *Ans.* $x^2 + (y - z)^2 = a^2$.

21. Find the cylinder with directing curve $x^2 = 1 - z$, $y = 0$ and having the line $x - y = 1$, $z = 0$ as one of its generators. Sketch the surface. *Ans.* $(x - y)^2 = 1 - z$.

22. Find the cylinder whose yz-trace is $z = y^3$ and which contains as a generator the line $x = y = z$. *Ans.* $(x - y)^3 = x - z$.

23. Find the cylinder whose zx-trace is $x^2 + 4z^2 = 4$ and which contains as a generator the line $3x - y + z = 1$, $8x - 3y + 2z = 2$.
Ans. $(2x - y)^2 + 4(2z + y)^2 = 16$.

24. What form is assumed by a cylinder when the directing curve becomes a straight line?

25. Derive the intercept form of the equation of the plane by considering the plane as a cylinder with xy-trace $\frac{x}{a} + \frac{y}{b} = 1$ and having the line $\frac{y}{b} + \frac{z}{c} = 1$, $x = 0$ as a generator.

26. A point moves so as to remain equidistant from the x-axis and the plane $y + z = a$. Find the equation of its locus. What kind of surface is generated? *Ans.* $2y^2 + 2z^2 = (y + z - a)^2$.

27. A point moves so that its distance from a fixed line is in a constant ratio e to its distance from a fixed plane, the line and plane being parallel. Prove that its locus is a quadric cylinder — elliptic, parabolic, or hyperbolic according as $e < 1$, $e = 1$, or $e > 1$. (Take as the fixed line, $x = ae$, $y = 0$; as the fixed plane, $x = \frac{a}{e}$.)

28. A point lies at the distance 1 from the line $x = 2$, $z = 3$. Find the equation of its locus.

29. A moving point is equidistant from the point (1, 0, 0) and the line $x + 1 = 0$, $y = 0$. Find the equation of its locus.
Ans. $z^2 = 4x$.

196. Cones. A *cone* is the surface generated by a moving line that always passes through a fixed point, called the *vertex*, and intersects a fixed curve, called the *directing curve.* It is the ruled surface all of whose generators pass through a fixed point. Like the cylinder, a cone may or may not be a quadric surface.

The equation of a cone with given vertex and directing curve may be found as in the following

Example: Find the equation of the cone with vertex at V: (0, 0, 1) and directing curve

$$x^2 + y^2 = 1, z = 0.$$

Let P: (x, y, z) be any point of the surface, and Q: $(x_Q, y_Q, 0)$ the point where the generator through P intersects the directing curve. The equations of the generator VQ in the symmetric form are

$$\frac{x}{x_Q} = \frac{y}{y_Q} = \frac{z-1}{-1},$$

whence

$$x_Q = \frac{-x}{z-1}, \quad y_Q = \frac{-y}{z-1}.$$

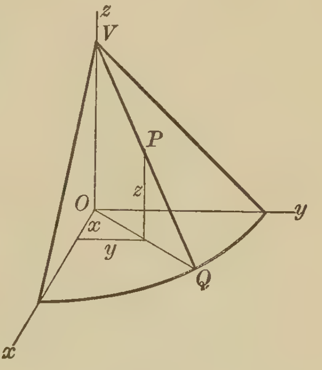


FIG. 139

But

$$x_Q^2 + y_Q^2 = 1.$$

Eliminating the parameters, we find

$$x^2 + y^2 = (z-1)^2.$$

197. The elliptic cone. Let the directing curve of a cone whose vertex is at the origin be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ z = c.$$

The equations of the generator OQ through P: (x, y, z) ar

$$\frac{x}{x_Q} = \frac{y}{y_Q} = \frac{z}{c};$$

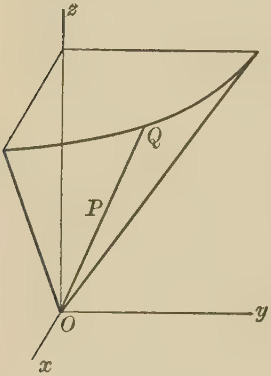


FIG. 140

also we have

$$\frac{x_Q^2}{a^2} + \frac{y_Q^2}{b^2} = 1.$$

Elimination of the parameter gives the equation

$$(9) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

From this equation we deduc the following properties:

(*a*) The surface has three plane and three lines of symmetry, an a center (the vertex).

(*b*) Sections parallel to the xy-plane are similar ellipse increasing in size as the cutting plane recedes from the vertex. The trace in the xy-plane is a point (i.e. a point ellipse).

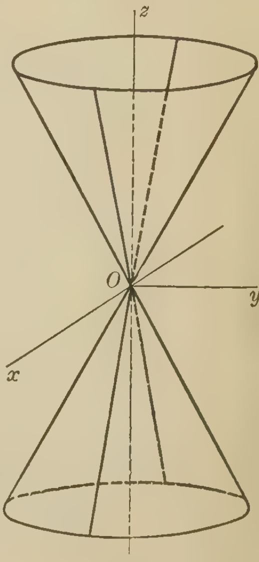


FIG. 141

(*c*) The sections parallel to the yz-plane, also those parallel to the zx-plane, are similar hyperbolas with transverse axis parallel to Oz, and increasing in size as the cutting plane recedes from the vertex. The traces in the yz- and zx-planes are pairs of lines.

(*d*) The surface consists of two sheets extending to infinity.

It is easy to prove that the equation of every cone whose directing curve is a conic section is of the second degree; further-more, by suitable coördinate transformations (analogous to those of Chapter VII) the equation of every quadric cone can be reduced to the form (**9**). Hence there is only a single species of quadric cone; from this one surface every kind of conic can be cut. Since the surface is most clearly visualized by means of its elliptic sections, it is usually called the *elliptic cone.*

That one of the lines of symmetry that passes through the center of the directing ellipse is called the *axis* of the cone, and sections by planes perpendicular to the axis are *right sections.*

When $a = b$, the right sections are circles, and the surface is the circular cone.

EXERCISES

1. Find the equation of the cone with vertex at O and directing curve $y^2 - z^2 = a^2$, $x = a$. Is this a cone of revolution?

Ans. $x^2 - y^2 + z^2 = 0.$

2. Find the equation of the cone with vertex (0, 4, 0), whose zx-trace is $\frac{x^2}{4} + \frac{z^2}{9} = 1$. Sketch the surface.

3. Find the equation of the cone with vertex at O, whose trace in the plane $z = 2$ is $y^2 = 8x$. *Ans.* $y^2 = 4zx.$

4. Find the equation of the cone with vertex $(a, 0, 0)$, whose yz-trace is the circle $y^2 + z^2 = 2ay$. Draw this cone.

Ans. $y^2 + z^2 + 2xy = 2ay.$

5. Find the equation of the cone with vertex (0, 0, 3), having as directing curve the parabola $y = 1 - x^2$, $z = 0$.

6. Find the equation of the cone having as vertex the point (1, 3, 5) and as xy-trace the circle $x^2 + y^2 = 4$.

7. Find the equation of the cone whose yz-trace is the curve $yz = 1$ and whose vertex is (4, 0, 0).

8. Find the equation of the cone with vertex at O whose directing curve is $y = x^4$, $z = 2$.

Classify, describe, and sketch the following surfaces.

9. $x^2 - 2y^2 - 2z^2 = 0.$

10. $\frac{y^2}{4} - \frac{x^2}{4} - \frac{z^2}{9} = 0.$

11. $y^2 + 2z = 0.$

12. $y^2 - z^2 = 16x.$

13. $4x^2 + y^2 + z^2 = 1.$

14. $y^2 + z^2 = 2x.$

15. $y^2 + z^2 = 2y.$

16. $x^2 + 3y^2 - 2z^2 = 0.$

17. $\frac{x^2}{4} + \frac{y^2}{100} + \frac{z^2}{1} = 1.$

18. $z^2 = x + y.$

19. $\frac{x^2}{4} + \frac{y^2}{9} = 1.$

20. $x^2 + y^2 - z^2 + 1 = 0.$

21. $x + y^2 + z^2 = 0.$

22. $x^2 - 2y^2 + z^2 = 1.$

23. What form is assumed by a cone when the directing curve becomes a straight line?

24. Derive the intercept form (§ 163) by considering the plane as a cone with vertex $(0, 0, c)$ and directing curve $\frac{x}{a} + \frac{y}{b} = 1$, $z = 0$.

25. What is represented by the equation (cylindrical coördinates) $r = kz$?

26. If the vertex and axis of a circular cone are given, together with the angle made by the generators with the axis, the equation of the cone can be found by using the formula for the angle between two lines (§ 141). Employ this method to find the equation of the circular cone with vertex at O, axis Oz, and generators making an angle of 30° with the axis. *Ans.* $3x^2 + 3y^2 = z^2.$

27. The vertex of a circular cone is at $(1, 0, 0)$, the axis is Ox, and the generators make with the axis the angle $\arccos \frac{1}{4}$. Find the equation of the cone by the method of Ex. 26.

28. Find the equation of a circular cone with vertex $(2, 1, 2)$ and axis $5x - 3y - 3z = 1$, $2x - 3y + 6z = 13$, if the generators make with the axis an angle of 45°.

Ans. $(3x + 4y + z - 12)^2 = 13[(x-2)^2 + (y-1)^2 + (z-2)^2].$

29. Find the equation of the cone generated by revolving the line $y = 3z + 4$, $x = 0$ about Oy.

30. Solve Ex. 29 by the method of Ex. 26.

31. By two methods, find the equation of the cone generated by revolving the line $x = y = z$ about Ox. *Ans.* $y^2 + z^2 = 2x^2.$

32. Find the equation of the circular cone with axis Oz, one of whose generators is the line $y = 2z$, $x = 3z$. Solve by two methods.

33. Find the equation of the circular cone with axis Oy, if one of the generators is the line $3x - y + z = 2$, $x + 2y - z + 4 = 0$. Solve by two methods.

34. Find the equation of the surface generated by a variable circle whose center moves along the line $2z = y$, $x = 0$, the circle always lying in a plane parallel to the zx-plane and intersecting the y-axis.
Ans. $x^2 + z^2 = yz$.

35. Find the equation of the surface generated by a variable circle parallel to the plane $z = 0$ and intersecting the lines $x = y = 0$ and $y = z$, $x = 0$.

36. Find the equation of the surface generated by a variable circle parallel to the yz-plane, intersecting the x-axis, and having its center on the circle $x^2 + y^2 = a^2$, $z = 0$.
Ans. $(y^2 + z^2)^2 + 4x^2y^2 = 4a^2y^2$.

37. A *conoid* is the surface generated by a line moving always parallel to a given plane and intersecting a fixed line and a fixed curve. Find the equation of the circular conoid generated by a line parallel to the xy-plane intersecting the line $y = h$, $x = 0$ and the circle $x^2 + z^2 = a^2$, $y = 0$. Draw this surface. *Ans.* $h^2x^2 = (a^2 - z^2)(h - y)^2$.

38. Find the equation of the conoid generated by a line parallel to the zx-plane intersecting the circle $x^2 + y^2 = a^2$, $z = 0$ and the line $y = z$, $x = 0$. *Ans.* $x^2y^2 = (y - z)^2(a^2 - y^2)$.

39. Find the equation of the parabolic conoid formed by a line parallel to the zx-plane intersecting the parabola $y^2 = 4ax$, $z = 0$ and the line $z = b$, $x = 0$. *Ans.* $y^2z - by^2 + 4abx = 0$.

40. Find the equation of the surface generated by a line parallel to the xy-plane, two of the traces being the parabolas $z^2 = x$, $y = 0$ and $z^2 = 2y$, $x = 0$. Prove that this surface is a cylinder.

41. Find the equation of the surface generated by a line parallel to the zx-plane intersecting the curves $y^2 + z^2 = a^2$, $x = 0$ and $y^2 = 4ax$, $z = 0$. *Ans.* $y^4z^2 = (y^2 - 4ax)^2(a^2 - y^2)$.

198. The general equation of the second degree.

The most general equation of the second degree in three variables has the form

$$(10)\quad Ax^2+By^2+Cz^2+Dyz+Ezx+Fxy+Gx+Hy+Iz+K=0.$$

By changing the directions of the coördinate axes, the origin remaining fixed, the terms in yz, zx, and xy can be removed. We will assume this to have been done, and will thus consider only the equation

$$(11)\qquad Ax^2 + By^2 + Cz^2 + Gx + Hy + Iz + K = 0.$$

Upon completing the squares in x, y, z, it is easily shown that this equation represents a quadric with its planes and axes of symmetry parallel to the coördinate planes and axes.

Example: Discuss the surface

$$4x^2 - 3z^2 + 8x + 12y + 12z + 16 = 0.$$

Completing the squares in x and z, we find

$$4(x^2 + 2x + 1) - 3(z^2 - 4z + 4) = -12y - 16 + 4 - 12,$$

or

$$\frac{(z-2)^2}{4} - \frac{(x+1)^2}{3} = \frac{y+2}{1}.$$

This surface is a hyperbolic paraboloid with vertex at $(-1, -2, 2)$ and axis parallel to Oy.

EXERCISES

Classify and describe the following surfaces.

1. $x^2 - y^2 - 2z^2 = 4x.$

2. $x^2 - y^2 - 2z^2 = y.$

3. $x^2 + y^2 = 2ay.$

4. $4x^2 + 4y^2 + 3z^2 - 8x - 4y + 2 = 0.$

5. $9y^2 - 4z^2 - 6y = 4z.$

6. $2x^2 - y^2 - z^2 + 4y + 6z - 13 = 0.$

7. $4y^2 + z^2 - 2x + 6y - 5z + 3 = 0.$

8. $x^2 - 3y^2 + 2z^2 - 4x - 3y - 2z = 0.$

9. $zx - 3x + 4z = 0.$

10. $x^2 + 3z^2 - y^2 - 2y = 1.$

11. Prove that a quadric surface is determined by nine points.

12. Find the equation of the quadric through the points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(2, 2, 2)$, $(2, 3, 4)$. Check the answer. What kind of surface is represented?

Ans. $3x^2 - 5y^2 + 2z^2 - 3x + 5y - 2z = 0.$

13. How many points are required to determine a quadric if the directions of its axes of symmetry are given?

14. Show that the locus of a point equidistant from a given line and a given point is a parabolic cylinder.

15. Find the equation of the surface generated by revolving the curve $x^2 = y^2 - 2ay$, $z = 0$ about Oy.

16. Find the equation of the surface generated by revolving the line $y = x$, $z = x + 1$ about Ox. *Ans.* $2x^2 - y^2 - z^2 + 2x + 1 = 0$.

17. Find the equation of the surface generated by revolving the line $y = a$, $z = 2x$ (*a*) about Oz; (*b*) about Ox.

CHAPTER XV

CURVES

199. Surfaces through a given curve. Let the equations of any curve be denoted by

$$u = 0, \; v = 0,$$

where u and v are certain expressions in x, y, and z. By reasoning analogous to that of § 166, we establish the

THEOREM: *The equation*

$$\mathbf{u + kv = 0} \tag{1}$$

represents, for all values of the constant k, a surface passing through the curve

$$u = 0, \; v = 0.$$

As noted in § 148, a curve may be represented analytically by the equations of any two of the infinite number of surfaces having that curve as their intersection. From the given pair of equations it is often possible to derive a simpler pair from which the form and properties of the curve are more readily seen*. One method of doing this is by suitable choice of k in formula (**1**); a second method, closely related to this one, will be developed in the next article.

200. Projecting cylinders. If a perpendicular be dropped from a point P to a plane, the foot P' of the perpendicular is called the *projection* of P upon the plane, and the line PP' is the *projecting line.* If all the points of a curve in space be projected upon any plane, their projections form a certain curve in that plane: this curve is called the *projection* of the given curve, and the cylinder whose generators are the projecting lines is the *projecting cylinder* of the curve upon the plane. Figure 142 shows the projection of a curve upon the xy-plane, together with that portion of the projecting cylinder that lies between the curve and its projection.

---

* Note, for example, the replacing of the general equations of a straight line by equations in the symmetric form.

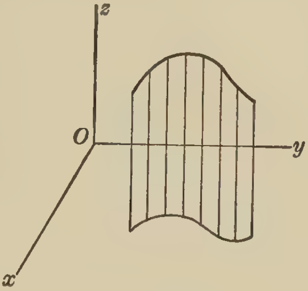


Fig. 142

Upon eliminating z between the equations of a curve we obtain a third equation which is satisfied whenever both of the original equations are satisfied, and therefore represents a surface through the given curve. Further, since the new equation does not contain z, it represents a cylinder perpendicular to the xy-plane — i.e. the xy-projecting cylinder of the given curve*. In this way we obtain the following rule:

To obtain the xy-projecting cylinder of a given curve, eliminate z between the equations of the curve; similarly for the other projecting cylinders.

An especially useful method of drawing a curve in space is to represent it as the intersection of two of its projecting cylinders.

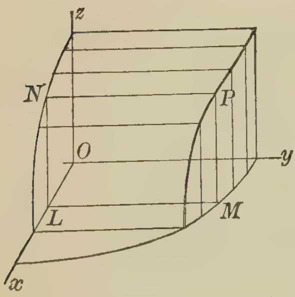


Fig. 143

Examples: (*a*) Find the projecting cylinders of the curve

$$4x^2 + y^2 + 3z^2 = 5,$$
$$2x^2 + y^2 + z^2 = 3,$$

and draw the curve.

Eliminating x by subtracting twice the second equation from the first, we get

$$y^2 - z^2 = 1$$

---

[*] A formal proof of this process is easily written out by the method of § 195.

as the equation of the yz-projecting cylinder. Similarly, the other cylinders are found to be

$$x^2 + z^2 = 1, \tag{3}$$

$$x^2 + y^2 = 2. \tag{4}$$

Thus one projection of the curve is an arc of a hyperbola, while the other two are circular arcs. In Fig. 143 the curve is represented as the intersection of the circular cylinders (3) and (4); since the figure is symmetric with respect to all three coördinate planes it is sufficient to draw the part lying in the first octant. The construction is as follows: through any point L on Ox draw lines parallel to Oy and Oz intersecting the directing curves of the cylinders at M and N. Through M and N draw the generators of the cylinders respectively parallel to Oz and Oy; their point of intersection P is a point of the curve.

(*b*) Find the projecting cylinders of the curve

$$x^2 + y^2 - z^2 = z,$$
$$x^2 + y^2 - z^2 = 1.$$

Subtracting the second equation from the first, we get

$$z = 1.$$

This is the equation of both the yz- and the zx-projecting cylinder. Substitution of this value of z in either of the original equations gives

$$x^2 + y^2 = 2.$$

Thus the curve is a right section of a circular cylinder: it is a circle of radius $\sqrt{2}$ with center in the z-axis, lying in the plane $z = 1$.

EXERCISES

1. Find the equation of a quadric surface through the curve $x^2 + y^2 + z^2 = 3, 3x^2 + y^2 - z^2 = 3$ and through the point $(1, 1, 2)$. Classify, describe, and sketch the three surfaces here occurring.

Ans. $2x^2 + y^2 = 3$.

2. Find the equation of a quadric through the curve $y^2 = 4ax$, $x^2 = 2yz$ and the point (a, a, a).

Ans. $3x^2 - y^2 - 6yz + 4ax = 0$.

3. Find the equation of a quadric through the curve $x^2 - y^2 = z$, $x^2 + z^2 = y$ and the point $(1, 1, -1)$. Classify the three surfaces.
Ans. $y^2 + z^2 - y + z = 0$.

4. If $u = 0$ and $v = 0$ are two quadrics, prove that $u + k v = 0$ is also a quadric. Are there any exceptions?

5. If $u = 0$ and $v = 0$ are two spheres, prove that $u + k v = 0$ is also a sphere. Are there any exceptions?

6. If $u = 0$ is a quadric and $v = 0$ a plane, what is represented by $u + k v = 0$?

7. Find the equation of a quadric through the curve $x^2 + y^2 - z^2 = 1$, $x + y + z = 3$ and the point $(3, 1, 1)$.

8. Classify and describe the surfaces occurring in examples (*a*) and (*b*), § 200.

In each of the following cases, classify the given surfaces, find the projecting cylinders of the curve of intersection, and draw the curve.

9. $y^2 = 4x$, $z = y$.

10. $y^2 + z^2 = a^2$, $x + y = a$.

11. $x^2 + y^2 + z^2 = 4a^2$, $x^2 + z^2 = a^2$.

12. $2x^2 + y^2 + 4z^2 = 5$, $x^2 + y^2 - 4z^2 + 2 = 0$.

13 $2x^2 + z^2 = y + 1$, $z^2 + y = 1$.

14. $x^2 + y^2 + z^2 = 4$, $x^2 - y^2 - 3z^2 + 4 = 0$.

15. $x^2 + y^2 + 2z^2 = 5$, $x^2 + y^2 + z^2 = 1$. *Ans.* No intersection.

16. $x^2 + y^2 = z + 1$, $x^2 - y^2 = z - 1$.

17. $x^2 + z^2 = 2y$, $y = x$.

18. $4x^2 + 3y^2 + 5z^2 = 4$, $x^2 + y^2 + z^2 = 1$.

19. $3x^2 + y^2 - 2z^2 = 6$, $x^2 - y^2 + 10z^2 = 2$.

20. $xy = 1$, $yz = 1$.

21. $2x^2 + y^2 - z^2 = 4$, $x^2 - y^2 = z^2$.

22. $x^2 + y^2 = z$, $x^2 + y^2 = z^2$.

23. $\frac{x^2}{3} - \frac{y^2}{1} - \frac{z^2}{1} = 1$, $\frac{x^2}{6} + \frac{y^2}{1} + \frac{z^2}{1} = 1$.

24. Find the equation of the surface generated by revolving the curve $x^2 + y^2 + 2z^2 = 1$, $y = x$ about Oy. *Ans.* $2x^2 + 2z^2 = 1$.

25. Find the equation of the surface generated by revolving the curve $2x^2 - y^2 + z^2 = 1$, $y = x$ about Oz. *Ans.* $x^2 + y^2 + 2z^2 = 2$.

26. Find the equation of the surface generated by revolving the curve $x^2 - y^2 = z - 1$, $x^2 = z$ about Oz. *Ans.* $x^2 + y^2 = z + 1$.

27. Find the equation of a cone with vertex at the origin and directing curve $y = x + a$, $x^2 + z^2 = a^2$. *Ans.* $x^2 + z^2 = (x - y)^2$.

28. Prove that the equations $2z^2 + 3x + z = 5$, $z^2 - y + z = 2$ represent cylinders, and show that these surfaces intersect in a plane curve.

29. Prove that the surfaces $x^2 + y^2 = z$, $x^2 + y^2 = 2x$ intersect in a plane curve. Draw the figure.

30. Prove that if the terms of the second degree in the equations of two quadrics are (or can be made) identical, the surfaces intersect in a plane curve.

201. Plane sections of a quadric surface. To investigate the intersection of a quadric surface and a plane, let us take the xy-plane parallel to the cutting plane, which of course involves no loss of generality. Thus we have to study the curve

$$(1)\quad Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy + Gx + Hy + Iz + K = 0,$$

$$(2)\quad z = k.$$

Substituting $z = k$ in (1), we get

$$(3)\quad Ax^2 + By^2 + Fxy + (Ek + G)x + (Dk + H)y + Ck^2 + Ik + K = 0,$$

which is a quadric cylinder through the given curve. Equations (2) and (3) taken together exhibit the curve as a right section of the cylinder (3); thus we have the

THEOREM: *Every plane section of a quadric surface is a conic.*

There is a limiting case, however, in which the section reduces to a single straight line; the student may investigate this case.

202. Plane sections of a quadric cylinder. To determine the curve of intersection of a plane and a quadric cylinder, let the xy-plane be taken perpendicular to the generators of the cylinder, so that the equation of the cylinder is

$$(1) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Further, let the yz-plane be perpendicular to the cutting plane, with the origin at the point of intersection of the cutting plane and the z-axis. Thus the cutting plane passes through the x-axis: let it be the plane $x\,Oy'$ (Fig. 144) making an angle α ($\neq 90°$) with the xy-plane. If P' is any point of the section and P its xy-projection, the distances $LP = y$ and $LP' = y'$ are connected by the relation

$$y = y' \cos \alpha.$$

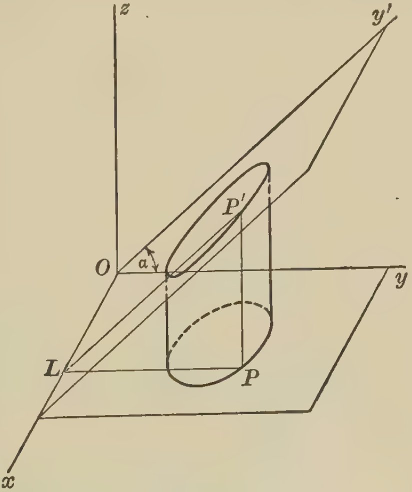


FIG. 144

Substituting this value of y in (1), we get

$$(2)\ Ax^2 + Bxy' \cos \alpha + Cy'^2 \cos^2\alpha + Dx + Ey' \cos \alpha + F = 0$$

as the equation of the section referred to Ox, Oy' as axes. Now the species of the conic (2) depends * upon the sign of the quantity $(B \cos \alpha)^2 - 4\,AC \cos^2 \alpha$, or $(B^2 - 4\,AC) \cos^2 \alpha$. This quantity is negative, zero, or positive according as $B^2 - 4\,AC$ is negative, zero, or positive. Thus we have the

THEOREM: *Every oblique plane section of a quadric cylinder is a conic of the same species as the right section.*

COROLLARY: *Every oblique plane section of a circular cylinder is an ellipse.*

---

* See § 87.

203. The conic sections. It was stated in § 66 that the various conics may be obtained as sections of a circular cone by a plane. To prove this, let us take the yz-plane perpendicular to the cutting plane, whence the equation of the latter may be written in the form

$$(1) \qquad z = m_1 y + k,$$

and generate the cone by revolving the line $z = my$, $x = 0$ about Oz, so that its equation is

$$(2) \qquad z^2 = m^2 (x^2 + y^2).$$

Eliminating z between (1) and (2), and rearranging, we get

$$m^2 x^2 + (m^2 - m_1{}^2) y^2 - 2 k m_1 y - k^2 = 0$$

as the equation of the xy-projecting cylinder of the curve in question. This cylinder is elliptic, parabolic, or hyperbolic according as m is (numerically) greater than, equal to, or less than m_1. Applying the result of § 202, we obtain the

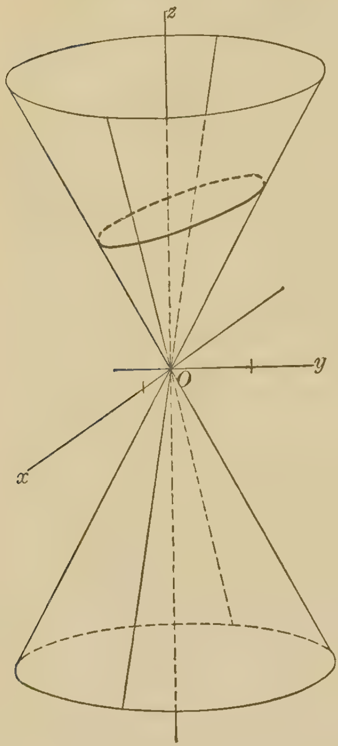


Fig. 145

Theorem: *A plane section of a circular cone is an ellipse, parabola, or hyperbola according as the angle between the axis of the cone and the cutting plane is greater than, equal to, or less than the angle between the axis and the generators of the cone.*

The way in which the various limiting forms of the conics are obtained may be investigated by the student in the exercises below.

EXERCISES

1. Prove that every plane section of a sphere is a circle. (Take the cutting plane parallel to a coördinate plane. Is the proof still general?)

2. Prove that two spheres intersect in a circle.

3. Find the center and radius of the circle $x^2 + y^2 + z^2 = 1$, $x + y + z = 1$. *Ans.* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; $\frac{1}{3}\sqrt{6}$.

4. Find the center and radius of the circle $x - y - z + 2 = 0$, $x^2 + y^2 + z^2 - 2x - 4z + 4 = 0$. *Ans.* $(\frac{2}{3}, \frac{1}{3}, \frac{7}{3})$; $\frac{1}{3}\sqrt{6}$.

5. Find the equation of the circular cylinder whose right section is the circle $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 4x$.

6. Find the equations of the tangent to the circle $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 + z^2 - 3x - y - 2z = 0$ at the point $(1, 2, 2)$.

7. Solve Ex. 6 by another method.

8. Find the equations of the circle generated by revolving the point $(3, 1, 2)$ about the line $x = 1$, $y = 2$.

9. Find the equations of the circle generated by revolving the point $(0, 0, 0)$ about the line $x = y$, $z = y - 3$.

Ans. $x + y + z = 0$, $x^2 + y^2 + z^2 = 2x + 2y - 4z$.

10. A point is at the distance 4 from the point $(2, 3, 0)$ and at the distance 1 from the plane $2x + y + 2z = 6$. Find the equations of its locus, and discuss the curve.

11. Prove that the locus of a point whose distances from a fixed point and a fixed plane remain constant consists of two circles.

12. Find the equation of the cone whose vertex is at O and whose directing curve is the circle of Ex. 3.

13. Solve Ex. 12 by the method of Ex. 26, p. 290.

14. A *great circle* of a sphere is a circle cut from the sphere by a plane through the center. Find the equations of that great circle of the sphere $x^2 + y^2 + z^2 = 9$ that passes through $(3, 0, 0)$, $(2, 1, -2)$.

15. Solve Ex. 14 by a second method.

16. Outline a method for finding the equations of a circle passing through three given points.

17. Find the equations of the circle through the points $(0, 2, -1)$, $(4, 0, -3)$, $(3, 2, -2)$.

Ans. $x^2 + y^2 + z^2 - 2x + 6z + 1 = 0$, $x - y + 3z + 5 = 0$.

18. Prove explicitly the corollary of § 202.

19. Prove that every plane section of an ellipsoid is an ellipse.

20. Prove analytically that the section of an elliptic paraboloid by a plane parallel to the axis is a parabola, and that all other plane sections are ellipses.

21. Prove analytically that the section of a hyperbolic paraboloid by a plane parallel to the axis is a parabola, and that all other plane sections are hyperbolas.

22. Find the equation of the cylinder whose right section is the ellipse $x^2 + z^2 = a^2$, $y = x$.

23. Find the center, semi-axes, and projecting cylinders of the ellipse $x^2 + z^2 = 2x$, $x^2 + z^2 = 2y$, and draw the curve.

24. Discuss the intersection of the surfaces $x^2 + y^2 = 1$, $y^2 + z^2 = 1$. Draw the figure.

25. Show by means of examples that two quadrics may intersect in a twisted curve, in two conics lying in different planes, or in a single conic.

26. Find the equation of a cone with vertex at O and the circle $x^2 + y^2 + a^2 = 0$, $z = c$ as directing curve.

27. What limiting form is approached by a circular cone with fixed directing curve when the vertex recedes indefinitely?

Show how to cut each of the following curves from a circular cone.

28. A circle.

29. A point circle.

30. A point ellipse.

31. An imaginary ellipse. (Cf. Ex. 26.)

32. A pair of real and parallel lines. (Cf. Ex. 27.)

33. A pair of imaginary parallel lines.

34. A pair of coincident lines.

35. A pair of intersecting lines.

36. An equilateral hyperbola.

Determine the species of conic cut from the cone $x^2 + y^2 = z^2$ by each of the following planes.

37. $x + y + z = 1$.

38. $x + y + 2z = 3$.

39. $4x + y + 5z = 10$.

40. $z = 3$.

41. $2x + 2z + 3 = 0$.

42. $4x - 3y + 5z = 5$.

43. $3x + 4y - 5z = 0$.

44. $3x - y + 2z = 6$.

INDEX

(The references are to pages.)

PLANE ANALYTIC GEOMETRY

Abscissa, 2, 3
Angle, between two curves, 79
  between two lines, 63
  bisectors, 61
  polar, 98
Area of a triangle, 12
Asymptotes, 200
  determination of —, 204
  equilateral hyperbola referred to —, 206
  of hyperbola, 120, 121, 200
  single equation representing pair of —, 201
Auxiliary circles, 186
Axis, of coördinates, 3
  of ellipse, 115
  of hyperbola, 120
  of parabola, 109
  of symmetry, 21
  polar, 98
  radical, 94

Center, of circle, 68
  of ellipse, 115
  of hyperbola, 120
  of symmetry, 22
  radical, 94
Chord, 79
  common — of two circles, 93
  of contact, 87, 163, 191
Circle, 68
  determined by three conditions, 72
  general equation, 69
  in polar coördinates, 99, 101
  limiting forms, 70
  standard equations, 68
  through intersection of two circles, 90
  through three points, 73
Composition of ordinates, 134
Conic sections, 105
  general equation, 107, 129
  in polar coördinates, 107
  limiting forms, 138, 139
  species of —, 106, 132, 197
  through five points, 142
Constants, 16
  arbitrary, 51
Coördinates, cartesian, 3
  polar, 98
Curve, 17, 20
  condition that point lies on —, 26
  family of curves, 51
  through intersection of two curves, 89
  tracing, 18, 23, 24, 99
  tracing by composition of ordinates, 134

Determinant, 12
Diameter, of a central conic, 191
  of a circle, 68
  of a conic, 167
  of a parabola, 167
Directrix, 105
  of ellipse, 114, 115
  of hyperbola, 119
  of parabola, 109
Discriminant of a quadratic equation, 76, 80
Distance, between parallel lines, 58
  between two points, 5
  of a point from a line, 58, 60

Eccentric angle, 186, 189
Eccentricity, 105
  of ellipse, 105, 114
  of hyperbola, 105, 119
  of parabola, 105, 109
  of rectangular hyperbola, 124
Ellipse, 106, 114, 133, 175
  general equation, 117, 133
  limiting forms, 116, 139
  standard equations, 114, 116
Equality, of geometric figures, 148
  of parabolas, 148

Equations, factorable, 27, 139
  incompatible, 31
  of first degree, 48
  of second degree, 106, 129
  quadratic, 75
  simultaneous, 29

Focus, 105
  of ellipse, 114, 115, 116
  of hyperbola, 119, 123
  of parabola, 109, 110, 111

Geometric constructions, 150, 177
Geometric properties, of the central conics, 180, 184, 193
  of the parabola, 161, 169

Hyperbola, 106, 119, 133, 177
  equilateral, 123, 133, 206
  general equation, 122, 133
  limiting form, 122, 138
  rectangular, 123, 133, 206
  standard equations, 119, 121

Infinity, curves intersecting at, 195
  points at —, 195
Intercepts on the axes, 23
Intersection, imaginary, 31
  number of points of —, 31
  of a curve with the axes, 23
  of two curves, 29

Latus rectum, 105
  of ellipse, 115
  of hyperbola, 119
  of parabola, 110
Line segments, directed, 1, 4
  equality of —, 2
Locus, as path of a moving point, 34
  by elimination of parameters, 164
  defined geometrically, 38
  of an equation, 17, 20

Midpoint of a line segment, 8

Normal, 156

Ordinate, 2
Origin of coördinates, 2, 3, 98

Parabola, 106, 109, 133, 146
  general equation, 112
  limiting form, 112, 139
  standard equations, 110, 111
  through four points, 144
Parallel lines, 10, 49, 112, 139
Parameter, 165, 187
Parametric equations, 187
  of the central conics, 188
Perpendicular bisector of a line segment, 35, 40
Perpendicular lines, 10, 49
Pole, 98

Quadrants, 4

Radius, of a circle, 68
  vector, 98
Roots, infinite, 65
  of a quadratic equation, 76
Rotation of axes, 126, 130

Scales, 4
Secant, 79
Similarity, 148
  of central conics, 174
  of parabolas, 148
Slope, of a curve, 79
  of a line, 9
Straight line, determined by two conditions, 53
  general equation, 48
  intercept form, 54
  normal form, 56
  parallel to an axis, 42
  point-slope form, 43
  slope form, 46
  through two points, 46, 47
Subnormal, 156
Subtangent, 156
Symmetry, 21, 22
  tests for —, 22, 25, 100

Tangent, 79
  at a given point of contact, 83, 154, 179
  having a given slope, 81, 159, 181
  length of —, 94, 96
  through an external point, 84, 159, 160
Transformation, of coördinates, 103, 125
  of equations, 28
Translation of axes, 125

Units, 4

Variables, 16
Vertex, of ellipse, 115
  of hyperbola, 119
  of parabola, 110

SOLID ANALYTIC GEOMETRY

Angle, between a line and a plane, 251
  between two lines, 218
  between two planes, 238
Axis, of coördinates, 208
  of revolution, 258
  of symmetry, 255

Center of symmetry, 255
Cone, 287
  circular, 259, 289, 300, 302
  elliptic, 287
Conic sections, 298, 300
  limiting forms, 302
Conoid, 291
Coördinate axes, 208
Coördinate planes, 208
Coördinates, cartesian, 208
  cylindrical, 221
Curve, 225
  as locus of a moving point, 229
  determination of points on a —, 226
  skew, 225
  twisted, 225
Cylinder, 282
  circular, 259
  perpendicular to a coördinate plane, 283
  plane sections of a quadric —, 299
  projecting, 294
  quadric, 284

Direction angles, 213
Direction cosines, 213, 214
  determination of —, 246, 247
  of a radius vector, 215
  of a line perpendicular to a plane, 232
  of the line through two points, 215
  proportional to three numbers, 214
Distance, between two planes, 235
  between two points, 210
  of a point from a plane, 235

Ellipsoid, 266
  limiting forms, 268
  of revolution, 259, 268
Equation, in two variables, 283
  of first degree, 231
  of second degree, 265, 291

Figures in space, 209

Generating curve, 258
Generator, 258, 277

Hyperboloid of one sheet, 268, 278
  of revolution, 259, 269, 272
Hyperboloid of two sheets, 270
  of revolution, 259, 271

Intercepts on the axes, 223
Intersection, of a curve and a surface, 227
  of three planes, 241
  of three surfaces, 227
  of two lines, 248
  of two surfaces, 225

Locus, of a moving point, 228
  of an equation, 222
  of two simultaneous equations, 225

Midpoint of a line segment, 210

Octants, 208
Origin of coördinates, 208

Paraboloid, elliptic, 273
  hyperbolic, 274, 279, 280
  of revolution, 259, 274
Parallel line and plane, 251
Parallel lines, 248
Parallel planes, 234
Perpendicular line and plane, 232, 250
Perpendicular lines, 219
Perpendicular planes, 238
Piercing points, 226
Plane, 230
  general equation, 231
  intercept form, 240
  normal form, 230
  parallel to a coördinate plane, 223
  perpendicular to a coördinate plane, 239
  projecting —, 243
  sections of a surface by a —, 223
  through a given line, 242
  through a given point, 236
  through three points, 236

Projection, of a broken line upon a line, 218
  of a curve upon a plane, 294
  of a line upon a plane, 243
  of a point upon a line, 218
  of a point upon a plane, 294

Radius vector, 215

Sphere, 259, 263
  line tangent to a —, 263
  plane tangent to a —, 263
Spheroid, oblate, 259, 261, 268
  prolate, 259, 268
Straight line, 242
  general equations, 242
  parallel to a coördinate plane, 247
  symmetric form, 245, 246, 247
Surface, 222, 255
  as locus of a moving point, 228
  determination of points on a —, 223
  generation by a moving curve, 258
  of revolution, 258
  quadric, 265
  ruled, 277
  sketching by parallel plane sections, 256
  through a given curve, 294
Symmetry, 255
  tests for —, 255

Torus, 259
Traces, 223